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On the splat singularity for the Muskat problem

Tesis presentada por Tania Pernas Castaño
para obtener el grado de Doctora en Matemáticas

Tesis doctoral dirigida por Diego Córdoba Gazolaz

Ó meu pai,
por animarme a perseguir os meus soños

Agradecimientos

Dicen que escribir una tesis es difícil, que requiere mucho trabajo y esfuerzo. Creo que es cierto y que sobre todo requiere motivación y constancia. Pero aquí me encuentro, con la tesis casi finalizada y con miedo a que estas palabras no sean capaces de reflejar todo lo que siento. Aun así, lo intentaré.

Comencemos por el principio:

Este trabajo no habría sido posible sin el apoyo y el estímulo de mi director Diego Córdoba Gazolaz. Sin conocerme de antemano confió en mi y me dio la oportunidad de embarcarme en esta experiencia académica.

Como todo el mundo sabe, el desarrollo de una tesis es una época de cambios y altibajos, que no sería posible de superar sin las personas con las que te vas encontrando en ella. Sin duda tengo que darle las gracias a Ángel Castro, por toda la paciencia, todo el tiempo, charlas y enseñanzas que me ha otorgado. No es solo uno de los mejores matemáticos que conozco, sino un fantástico hermano mayor. Gracias Ángel, por arrojar luz cuando más lo necesitaba.

Pero si pienso en charlas matemáticas, indudablemente hay más personas que se me vienen a la cabeza: Javi Gómez, siempre dispuesto a escuchar mis dudas; Alberto Martín, amigos inseparables de la biblioteca de Princeton; Nastasia Grubic, que me ha ayudado con nuevos problemas; Ángel David Martínez, que siempre tiene un hueco para mi y consigue transmitirme su motivación e interés por las matemáticas o Diego Alonso, que ha dedicado horas y horas a entender mis problemas y a apoyarme en todos mis avances.

Irremediablemente todas estas charlas o simplemente conversaciones en la sala común, cafés compartidos, visitas a otros despachos...han desembocado en amistades que no olvidaré y que han supuesto una ayuda a lo largo de esta etapa tan importante de mi vida.

Cuando echo la vista atrás y recuerdo el primer día en el que me incorporé, creí que me estaba metiendo en un mundo de “raros”, pero resultó que esos “raros” me acogieron mejor de lo que podría haber imaginado. Primero me encontré con Jeza, única chica del pasillo y prácticamente del instituto, que resultó ser una de mis primeras mejores amigas en Madrid. Gracias por todos los momentos que pasamos juntas tanto dentro como fuera de la pista de baile. En esos primeros días también estaba Alberto. ¿Quién iba a imaginar que seríamos tan amigos? Aunque ya no eres mi hermano académico, ni mi compañero de piso, me encanta que continúen esos cafés. Y, por supuesto, gracias a la asociación ASI SEASE. Todas las cenas, salidas, bailes y Eventos Alternativos “casi” Siempre Exitosos, que me sirvieron de desconexión. Gracias Cédric por tu inagotable energía y buen humor, a Miguel porque siempre consigues sacarme una sonrisa y Dani por escuchar mis lloriqueos.

Pasado el tiempo inicial de adaptación, te das cuenta de que tu lugar de trabajo pasar a ser como tu casa y, por supuesto, tus compañeros de despacho, tu familia. Cuantas vivencias, emociones y sentimientos que ocurren dentro de esas cuatro paredes. En el 412 se puede decir que ocurre de todo: hacemos manualidades, nos informamos de todas las noticias interesantes (y no tanto) del mundo, cotilleamos, recibimos visitas, discutimos, gruñimos, se podría decir que hasta hacemos deporte; y, bueno, de vez en cuando matemáticas. Gracias "compis" de despacho por aguantar mis agobios, cháchara, bromas, risas e incluso lloros. Jorge, muchas gracias por ayudarme a relativizar los problemas; Juanjo, por tu paciencia y por tener siempre unas palabras amables hacia mí y Víctor, por tus gruñidos y por amenizar nuestras charlas.

Aunque mi mayor tiempo lo pasé en el 412 no me olvido del 513, mi despacho de la UAM. Gracias chicos por acogerme tan bien. Diego, un placer conocerte y tenerte como compañero de despacho y casi de piso. Carlos, gracias por tu ayuda y compañerismo con la organización del Seminario Junior. Marta, gracias por tus consejos y charlas de chicas.

Ya adaptada a la vida de investigadora, te encuentras con que tu círculo de amistades ha aumentado casi sin darte cuenta. Solamente es necesario probar la comida de cada uno de los restaurantes y cantinas que hay en el campus. Esto une tanto, que ha supuesto que conozca y tenga por amigos a gente increíble. Gracias Eric por estar siempre con ganas y fuerzas para todo, por animarme y tenderme la mano tanto en los momentos buenos como malos. Gracias Ángel, por tu risa (aunque me pelee por ella), por preocuparte tanto por mi e implicarte siempre que puedes. Carlos, por ilustrarnos con tus datos interesantes, las comidas no serían lo mismo sin ellos. Cristina, siempre te animas a todos los disparates que se nos ocurren, no pudimos hacernos famosas pero aun estamos a tiempo, gracias por tu alegría y locura. Mari Ángeles, me alegro mucho de haberte conocido, siempre tienes palabras de aliento para todo el mundo, a pocas personas conozco tan buenas como tú, gracias por tu amistad y cariño.

Hay muchas personas, tanto en el ICMAT como en la UAM, que han estado presentes en esta andadura de una forma u otra. Iason, compañero de organización; Ángela, increible sucesora; Omar, siempre sonriente. Algunos que ya no están por aquí como: David Fernández, Leo, Elisa,...y así podría seguir hasta el infinito. Gracias a todos. Además no me puedo olvidar de mi familia adoptiva de Madrid. Chicky, Berto, Marta y Lauri, gracias por acogerme y tratarme como a una más.

Haciendo reflexión sobre esta etapa de mi vida, no puedo estar más contenta y agradecida. Cuando uno se plantea cómo le va a ir en una nueva ciudad en un tiempo de cambio, no es capaz de imaginar los distintos caminos que pueden surgir, yo no me lo imaginaba. Diego, gracias por estar ahí, por preocuparte por mi, por cuidarme, por creer en mi incluso cuando yo no podía hacerlo, por hacer que todo merezca la pena y sobre todo por acompañarme en la vida. Te quiero.

Inevitablemente todas las vivencias nuevas llevan a un entorno nuevo, pero toda mi energía y motivación no hubiese estado completa sin mis raíces. Muchas gracias a mis Chuliños y Peitudiñas, que me quieren aunque pase tiempo sin verlos. A Aita, Ángel y Marco, por quererme, por hacerme madrina y por apoyarme en todo.

Y, por supuesto, ¿quién sería yo sin mi familia? Gracias a Ángeles, Jose, Gloria y a la Abuela por creer en mi y en mi capacidad, dándome fuerzas para salir siempre adelante. Gracias Fran, Pili y Marta, por cuidarnos y por vuestro apoyo. Gracias a mi Tío Toñi, por cuidarme a pesar de la distancia. Y a mis padrinos, por tratarme como si fuese de la familia. Pero especialmente, mi pilar y mis incondicionales, Chus y Pitu. Pocas madres se desviven tanto por sus hijas y pocas hermanas son tan buenas. Estoy orgullosísima de vosotras y muy agradecida de teneros para mí. Os quiero muchísimo, todo esto no hubiese sido posible sin vuestro apoyo.

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Introducción - Resumen y conclusiones

El problema de Muskat se encuadra en el contexto de la física matemática, en concreto, en el estudio de las ecuaciones en derivadas parciales que provienen de la mecánica de fluidos. Rama que estudia el movimiento de los fluidos así como las fuerzas que lo provocan, o las interacciones entre el fluido y el contorno que los limita.

En este área es necesario hacer mención de las ecuaciones de Navier-Stokes. Son las ecuaciones que se utilizan para aproximar el movimiento de fluidos usuales como el agua, el aire o el aceite, y por ello son un modelo básico en diversas ciencias como la aeronáutica, la meteorología, la hidráulica, etcétera.

Por lo tanto, se puede considerar a las ecuaciones de Navier-Stokes como la ley fundamental que, junto las leyes de conservación de la masa, permite describir el movimiento de un fluido a partir de unas condiciones iniciales y de contorno determinadas.

Su expresión viene dada por:

$$\begin{cases} \rho(\frac{\partial u}{\partial t} + (u \cdot \nabla)u) = -\nabla p + \mu \Delta u + f, \\ \nabla \cdot u = 0, \\ \rho_t + (u \cdot \nabla)\rho = 0, \end{cases}$$

donde la condición $\nabla \cdot u = 0$ determina la incompresibilidad del flujo.

Estas ecuaciones, sin embargo, no rigen la dinámica de fluidos en medios porosos, donde el fluido se mueve por huecos (o poros) de una estructura sólida y ha de tenerse en cuenta la resistencia ofrecida por esta. Este tipo de medio es el que nos concierne en esta tesis. Existen muchas sustancias naturales como pueden ser rocas, suelos (por ejemplo: acuíferos y sedimentos petrolíferos), zeolitas, tejidos biológicos (como huesos, madera y corcho) o materiales hechos por el hombre tales como el cemento y las cerámicas, que pueden considerarse medios porosos.

Los fluidos en medios porosos son, por tanto, de gran interés en diversos problemas reales que abarcan muchas áreas de la ciencia aplicada y la ingeniería: filtración, mecánica (acústica, geomecánica, mecánica de suelos, mecánica de rocas), ingeniería (petrolífera, biorremediación), geociencias (hidrogeología, geofísica), biología y biofísica, ciencia de los materiales, etc.

Dada la inmensa concurrencia de los fluidos en estos medios y que las ecuaciones de Navier-Stokes no nos permiten modelar este fenómeno, como hemos comentado, se plantea el problema de cómo proceder de forma alternativa para la modelación de esta dinámica de forma eficaz.

Para cubrir esta necesidad apareció, la *ley de Darcy*, cuyo nombre se debe al ingeniero de puentes y caminos Henry Darcy (1803-1858), uno de los encargados del diseño y construcción del sistema de abastecimiento de agua potable de la ciudad de Dijon. En torno a 1850, Darcy descubrió esta ley experimental que describe adecuadamente la dinámica del flujo de un fluido incompresible en un medio poroso:

$$\frac{\mu}{\kappa}u = -\nabla p - (0, g\rho),$$

donde $(x,t) \in \mathbb{R}^2 \times \mathbb{R}^+$, $u = (u_1(x,t), u_2(x,t))$ es la velocidad del fluido incompresible ($\nabla \cdot u = 0$), $p = p(x,t)$ es la presión, $\mu = \mu(x,t)$ es la viscosidad dinámica del fluido, $\kappa = \kappa(x)$ es la permeabilidad del medio, $\rho = \rho(x,t)$ es la densidad del fluido y g es la aceleración debido a la gravedad.

Con esta nueva ley nuestro modelo para fluidos en medios porosos queda descrito por:

$$\begin{cases} \rho_t + (u \cdot \nabla)\rho = 0, \\ \frac{\mu}{\kappa}u = -\nabla p - (0, g\rho), \\ \nabla \cdot u = 0. \end{cases} \tag{1}$$

Estas ecuaciones son las que nos van a permitir estudiar el problema central de este trabajo, el *problema de Muskat.*

Morris Muskat (1906-1998) fue un ingeniero petrolífero estadounidense que, en colaboración con Milan W. Meres, utilizó la ley de Darcy para el estudio del flujo multifásico de agua, petróleo y gas en un yacimiento petrolífero.

Concretamente, el sistema (1) fue estudiado en [27] donde se modela la evolución de la interfase entre dos fluidos inmiscibles de diferente naturaleza en un medio poroso.

Se trata de un problema de frontera libre 2-dimensional, cuya interfase es causada por la discontinuidad entre las viscosidades y/o densidades de los fluidos:

$$(\mu, \rho)(x,t) := \begin{cases} (\mu^1, \rho^1) & x \in \Omega^1(t) \\ (\mu^2, \rho^2) & x \in \Omega^2(t) = \mathbb{R}^2 - \Omega^1(t) \end{cases}$$

donde μ^1, ρ^1, μ^2 y ρ^2 son constantes.

En este trabajo comenzaremos considerando un medio poroso en el cual la permeabilidad permanece constante, lo que se conoce como *problema homogéneo.*

En estas condiciones y asumiendo que nuestro medio es periódico en la variable horizontal, las ecuaciones de evolución se obtienen como sigue:

Buscamos una parametrización de la frontera libre mediante una curva,

$$\partial\Omega^j = \{z(\alpha,t) = (z_1(\alpha,t), z_2(\alpha,t)) : \alpha \in \mathbb{R}\} \quad \text{para } j = 1, 2$$

tal que la siguiente condición de periodicidad se cumpla

$$(z_1(\alpha + 2k\pi, t), z_2(\alpha + 2k\pi, t)) = (z_1(\alpha,t) + 2k\pi, z_2(\alpha,t))$$

con dato inicial, $z(\alpha, 0) = z_0(\alpha)$.

Gracias a la ley de Darcy podemos concluir que el fluido es irrotacional, es decir, $\omega = \nabla \times u = 0$, en el interior de cada dominio Ω^j para $j = 1, 2$ y que la vorticidad está concentrada en la frontera libre $z(\alpha,t)$ por una distribución de Dirac:

$$\omega(x,t) = \nabla^{\perp} \cdot u(x,t) = \varpi(\alpha,t)\delta(x - z(\alpha,t)), \tag{2}$$

donde $\varpi(\alpha,t)$ representa la amplitud de la vorticidad.

Debido a la incompresibilidad existe una función de corriente $\psi(x,t)$ tal que $u = \nabla^{\perp}\psi$. Si aplicamos el rotacional, como $\nabla \times \nabla^{\perp} = \Delta$, obtenemos así $\omega = \Delta\psi$. La teoría del potencial permite describir la solución de esta ecuación de Poisson como

$$\psi(x,t) = \frac{1}{2\pi}\int_{\mathbb{R}^2} \ln|x-y|\omega(y,t)dy.$$

Por tanto, aplicando $\nabla^{\perp} = (-\partial_{x_2}, \partial_{x_1})$ y (2) tenemos que

$$u(x,t) = \frac{1}{2\pi}\int_{\mathbb{R}^2} \frac{(x-z(\beta,t))^{\perp}}{|x-z(\beta,t)|^2}\varpi(\beta,t)d\beta.$$

Esta expresión es análoga a la ley de Biot-Savart para el campo magnético de un cable (c.f. [25]).

Tomando límites en la dirección normal a la frontera de la velocidad obtenemos que

$$u^{+}(z(\alpha,t),t) = \lim_{\varepsilon\to 0^{+}} u(z(\alpha,t)+\varepsilon\partial_{\alpha}^{\perp}z(\alpha,t)) = BR(z,\varpi)(\alpha,t) - \frac{\varpi(\alpha,t)}{2|\partial_{\alpha}z(\alpha,t)|^2}\partial_{\alpha}z(\alpha,t), \tag{3}$$

$$u^{-}(z(\alpha,t),t) = \lim_{\varepsilon\to 0^{-}} u(z(\alpha,t)+\varepsilon\partial_{\alpha}^{\perp}z(\alpha,t)) = BR(z,\varpi)(\alpha,t) + \frac{\varpi(\alpha,t)}{2|\partial_{\alpha}z(\alpha,t)|^2}\partial_{\alpha}z(\alpha,t), \tag{4}$$

donde $BR(z,\varpi)(\alpha,t)$ se conoce como integral de Birkhoff-Rott:

$$BR(z,\varpi)(\alpha,t) = \frac{1}{2\pi}PV\int_{\mathbb{R}} \frac{(z(\alpha,t)-z(\beta,t))^{\perp}}{|z(\alpha,t)-z(\beta,t)|^2}\varpi(\beta,t)d\beta.$$

Puesto que la frontera libre se mueve con el fluido, $z(\alpha,t)$ evolucionará con el campo de velocidades $u(x,t)$:

$$z_t(\alpha,t)\cdot\partial_{\alpha}^{\perp}z(\alpha,t) = u^{+}(z(\alpha,t),t)\cdot\partial_{\alpha}^{\perp}z(\alpha,t) = u^{-}(z(\alpha,t),t)\cdot\partial_{\alpha}^{\perp}z(\alpha,t).$$

Por tanto, es suficiente considerar la ecuación

$$z_t(\alpha,t) = BR(z,\varpi)(\alpha,t) + c(\alpha,t)\partial_{\alpha}z(\alpha,t)$$

para alguna función c. Nótese que diferentes funciones c se corresponden con diferentes parametrizaciones de la curva. Nosotros tomaremos:

$$\begin{aligned} c(\alpha,t) =& \frac{\alpha+\pi}{2\pi}\int_{\mathbb{T}} \frac{\partial_{\beta}z(\beta,t)}{|\partial_{\beta}z(\beta,t)|^2}\cdot\partial_{\beta}BR(z,\varpi)(\beta,t)d\beta \\ &- \int_{-\pi}^{\alpha} \frac{\partial_{\beta}z(\beta,t)}{|\partial_{\beta}z(\beta,t)|^2}\cdot\partial_{\beta}BR(z,\varpi)(\beta,t)d\beta. \end{aligned}$$

Esta elección permite eliminar la dependencia en α del módulo del vector tangente $\partial_{\alpha}z(\alpha,t)$ (para más detalles véase [11]), es decir,

$$|\partial_{\alpha}z(\alpha,t)|^2 = A(t).$$

Finalmente para cerrar nuestro sistema, usaremos de nuevo la ley de Darcy. Multiplicando por

$\partial_\alpha z(\alpha,t)$, es fácil ver que:

$$\varpi(\alpha,t) = -2\frac{\mu^1-\mu^2}{\mu^1+\mu^2}BR(z,\varpi)(\alpha,t)\cdot\partial_\alpha z(\alpha,t) - 2\kappa g\frac{\rho^2-\rho^1}{\mu^1+\mu^2}\partial_\alpha z_2(\alpha,t). \tag{5}$$

Aquí hemos usado $p^1(z(\alpha,t)) = p^2(z(\alpha,t))$ donde p^j denota la presión en Ω^j (véase [11]). Entonces,

$$\begin{aligned}(\mu^2u^- - \mu^1u^+)\cdot\partial_\alpha z &= -(\nabla p^2 - \nabla p^1)\cdot\partial_\alpha z - g(\rho^2-\rho^1)\partial_\alpha z_2\\ &= -\partial_n(p^2-p^1) - g(\rho^2-\rho^1)\partial_\alpha z_2 = -g(\rho^2-\rho^1)\partial_\alpha z_2.\end{aligned}$$

Usando (3) y (4), obtenemos (5).

Por lo tanto, la interfase del problema de Muskat queda descrita por:

$$(P)\begin{cases} z_t(\alpha,t) = BR(z,\varpi)(\alpha,t) + c(\alpha,t)\partial_\alpha z(\alpha,t),\\ BR(z,\varpi)(\alpha,t) = \frac{1}{2\pi}PV\int\frac{(z(\alpha,t)-z(\beta,t))^\perp}{|z(\alpha,t)-z(\beta,t)|^2}\varpi(\beta,t)d\beta,\\ c(\alpha,t) = \frac{\alpha+\pi}{2\pi}\int_{\mathbb{T}}\frac{\partial_\beta z(\beta,t)}{|\partial_\beta z(\beta,t)|^2}\cdot\partial_\beta BR(z,\varpi)(\beta,t)d\beta - \int_{-\pi}^{\alpha}\frac{\partial_\beta z(\beta,t)}{|\partial_\beta z(\beta,t)|^2}\cdot\partial_\beta BR(z,\varpi)(\beta,t)d\beta,\\ \varpi(\alpha,t) = -2\frac{\mu^1-\mu^2}{\mu^1+\mu^2}BR(z,\varpi)(\alpha,t)\cdot\partial_\alpha z(\alpha,t) - 2\kappa g\frac{\rho^2-\rho^1}{\mu^1+\mu^2}\partial_\alpha z_2(\alpha,t).\end{cases}$$

Para proceder con su estudio será necesario tener en cuenta dos condiciones sobre el problema: la primera de ellas es la que se conoce como condición cuerda-arco. Esta condición viene dada por la función

$$\mathcal{F}(z)(\alpha,\beta,t) = \frac{\beta^2}{|z(\alpha)-z(\alpha-\beta)|^2}, \quad \alpha,\beta\in\mathbb{R}$$

con

$$\mathcal{F}(z)(\alpha,0,t) = \frac{1}{|\partial_\alpha z(\alpha,t)|^2}.$$

Diremos que la condición cuerda-arco se satisface cuando $\mathcal{F}(z)\in L^\infty$. Esto se traduce en que la parametrización será aceptable y la interfase no presentará autointersecciones.

Por otro lado, debemos considerar la condición de Rayleigh-Taylor, que determina la estabilidad del problema. Rayleigh [29] y Saffman-Taylor [30] vieron que esta condición se debía satisfacer para que el modelo linealizado tenga soluciones locales en tiempo. Consiste en que la componente normal del salto de gradientes de la presión en la interfase tiene que tener signo:

$$\sigma(\alpha,t) = -(\nabla p^2(z(\alpha,t)) - \nabla p^1(z(\alpha,t)))\cdot\partial_\alpha^\perp z(\alpha,t) > 0,$$

donde p^j es la presión en Ω^j para $j=1,2$. Esta condición es equivalente a

$$\sigma(\alpha,t) = \frac{\mu^2-\mu^1}{\kappa}BR(z,\varpi)(\alpha,t)\cdot\partial_\alpha^\perp z(\alpha,t) + g(\rho^2-\rho^1)\partial_\alpha z_1(\alpha,t) > 0.$$

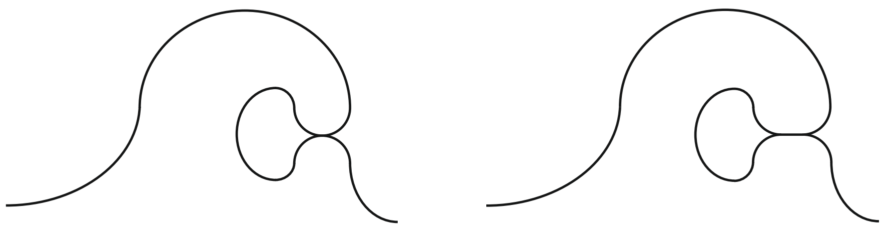
Para más detalles consúltese [11]. Usando el lema de Hopf se puede comprobar que esta condición se satisface siempre que $\mu^1=\rho^1=0$, como se demuestra en la sección 1.2. Para el caso con mismas viscosidades esta condición se cumple cuando el fluido más denso está bajo la interfase (véase [5]).

Bajo estas condiciones, para el caso más general (es decir, distintos valores de viscosidad y densidad) se demuestra en [11] existencia local en tiempo en espacios de Sobolev (H^k para $k\geq 3$). Previamente, considerando $\mu^1=\mu^2$ y la interfase como un grafo, se prueba existencia local en

[13]. Para la existencia local en el caso unifásico, es decir, $\mu^1 = \rho^1 = 0$, con dato inicial en H^2 se puede consultar [6]. Cuando consideramos mismas viscosidades, es decir, la interfase se forma por los distintos valores de las densidades, es conocido la existencia global para dato inicial pequeño y mediano en el régimen estable, c.f. [31], [8], [7], [9], [19] y [23]. Además existen datos iniciales con $\sigma > 0$ que en tiempo finito pasan a un régimen inestable en el que $\sigma < 0$, véase [5] y [22], y que más tarde, en tiempo finito, desarrollan una singularidad (véase [2]).

Esta tesis está enfocada, principalmente, en el estudio de singularidades. En particular, los dos tipos de singularidades que nos van a interesar son las introducidas para el problema de "water waves" en [4]: "splash" y "splat". Estas singularidades son descritas en el caso de la evolución de la frontera libre de una región de agua en el vacío. Para el estudio de este tipo de singularidades en el problema de Muskat, consideraremos el caso *unifásico*, esto es, un único fluido en el vacío con $\mu^1 = \rho^1 = 0$.

Grosso modo, las singularidades "splash" (véase Figura 1(a)) corresponden al caso en el que la interfase del fluido colapsa de forma suave sobre sí misma en un punto (una definición rigurosa se puede encontrar en [4]). Este tipo de singularidades ocurren en el problema de Muskat unifásico como se ha probado en [3]. Sin embargo, si se considera el caso de tres fluidos con diferentes densidades (bien ordenadas) y mismas viscosidades, no se pueden desarrollar singularidades "splash" (véase [21]).



(a) Singularidad de tipo "splash" (b) Singularidad de tipo "splat"

Figure 1: Singularidades a tiempo finito

El segundo tipo de singularidades se denominan, "splat" (Figura 1(b)). Esta formación es una variante de las singularidades "splash" en la cual la intersección es un arco de curva en lugar de un punto. Este tipo de singularidades también existen para el caso de "water waves", véase [4], y para el caso de "water waves" con vorticidad, como se puede véase en [16]. Sin embargo, en este trabajo vamos a demostrar que no se pueden formar singularidades de tipo "splat" para Muskat unifásico (Capítulo 1). El objetivo principal del Capítulo 1 es la demostración del siguiente teorema:

Theorem 0.0.1. *Sea $z_0(\alpha) \in H^k(\mathbb{T})$ para $k \geq 4$ y $\mathcal{F}(z_0)(\alpha, \beta) \in L^\infty$. Entonces el problema de Muskat (P) no desarrollará una singularidad de tipo "splat", es decir, no existirá un tiempo t para el cual dos intervalos disjuntos $I_1, I_2 \in \mathbb{R}$ cumplan $z(I_1, t) = z(I_2, t)$.*

Para el estudio de este tipo de singularidades consideraremos el problema de Muskat unifásico, es decir, $\mu^1 = \rho^1 = 0$.

$$(\mu, \rho)(x, t) = \begin{cases} (0,0) & \text{for } x \in \mathbb{R}^2 - \Omega(t), \\ (\mu^2, \rho^2) & \text{for } x \in \Omega(t), \end{cases}$$

donde $\Omega(t)$ es la región que contiene el fluido.

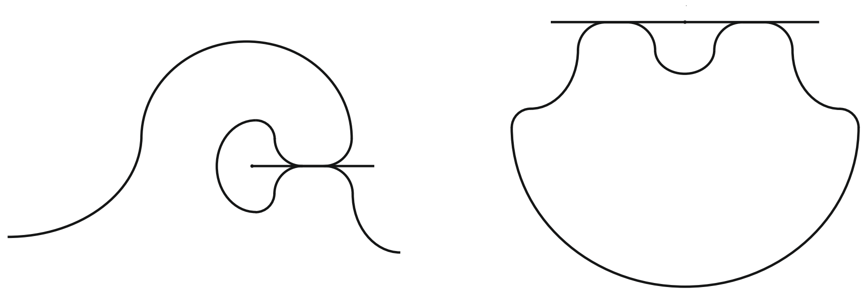
La idea de la demostración de este teorema sigue el siguiente razonamiento:

Supongamos que se forma una singularidad de tipo "splat" a tiempo T, es decir, la interfase $z(\alpha,t)$ colapsa sobre si misma en un arco de cuerda a tiempo $t=T$.

Si partimos de una curva que inicialmente es regular en nuestro dominio, concretamente $H^k(\mathbb{T})$ para $k\geq 4$, veremos que esta se vuelve analítica de forma instantánea. Además tendremos control sobre esa analiticidad siempre y cuando la regularidad H^k de la curva y la condición cuerda-arco no degeneren. Pero en nuestro dominio inicial Ω, a tiempo T, la condición cuerda-arco no se satisface. Por lo tanto no podemos garantizar la analiticidad en ese tiempo. Para resolver este inconveniente, transformamos nuestro dominio mediante la aplicación conforme P:

$$P(w)=(\tan(\frac{w}{2}))^{\frac{1}{2}},$$

donde la discontinuidad de la raíz se elige adecuadamente para deshacer el "splat". Esta aplicación cambia nuestro dominio Ω en otro $\tilde{\Omega}$ como se puede observar en la figura 2, separando los puntos de intersección.



(a) Dominio $\Omega(T)$

(b) Dominio $\tilde{\Omega}(T)$

Figure 2: Transformación del dominio mediante P

Las ecuaciones del problema de Muskat unifásico homogéneo en el nuevo dominio son (véase [3] para más detalles):

$$\tilde{z}_t(\alpha,t)=Q^2(\alpha,t)BR(\tilde{z},\tilde{\varpi})(\alpha,t)+\tilde{c}(\alpha)\partial_\alpha\tilde{z}(\alpha,t)$$

donde

$$Q^2(\alpha,t)=|\frac{dP}{dw}(z(\alpha,t))|^2=|\frac{dP}{dw}(P^{-1}(\tilde{z}(\alpha,t)))|^2,$$

$$\tilde{\varpi}(\alpha,t)=-2BR(\tilde{z},\tilde{\varpi})(\alpha,t)\cdot\partial_\alpha\tilde{z}(\alpha,t)-2\frac{\rho^2}{\mu^2}\partial_\alpha(P_2^{-1}(\tilde{z}(\alpha,t)))$$

y

$$\begin{aligned}\tilde{c}(\alpha,t)&=\frac{\alpha+\pi}{2\pi}\int_{\mathbb{T}}\frac{\partial_\beta\tilde{z}(\beta,t)}{|\partial_\beta\tilde{z}(\beta,t)|^2}\cdot\partial_\beta BR(\tilde{z},\tilde{\varpi})(\beta,t)d\beta\\&-\int_{-\pi}^{\alpha}\frac{\partial_\beta\tilde{z}(\beta,t)}{|\partial_\beta\tilde{z}(\beta,t)|^2}\cdot\partial_\beta BR(\tilde{z},\tilde{\varpi})(\beta,t)d\beta.\end{aligned}$$

La condición de Rayleigh-Taylor en términos de $\tilde{z}$ es

$$\tilde{\sigma}(\alpha,t)=\frac{\mu^2}{\kappa}BR(\tilde{z},\tilde{\varpi})(\alpha,t)\cdot\partial_\alpha^\perp\tilde{z}(\alpha,t)+\rho^2 g\nabla P_2^{-1}(\tilde{z}(\alpha,t))\cdot\partial_\alpha^\perp\tilde{z}(\alpha,t).$$

En este nuevo dominio existe una solución del problema transformado, que llamaremos $\tilde{z}(\alpha,t)$, definida para $0<t\leq T$. Probaremos, gracias a que $\tilde{\sigma}>0$ para $0\leq t\leq T$ y que en $\tilde{\Omega}$ se cumple la condición cuerda-arco, que $\tilde{z}$ es analítica para $0\leq t\leq T$; donde $\tilde{\sigma}$ es la condición de Rayleigh-Taylor en $\tilde{\Omega}$ y que no cambia de signo con respecto a σ. Por tanto, la curva z es analítica en $[0,T]$, en particular a tiempo T, llegando así a una contradicción y dejando demostrada la no existencia de singularidades "splat".

El trabajo realizado en este capítulo ha sido publicado en un artículo científico en Transactions of the American Mathematical Society (véase [14]).

Nuestro escenario hasta este momento ha sido un medio poroso cuya permeabilidad permanece constante. Pero podríamos preguntarnos ¿qué ocurre si esta varía?

Para el caso en el que la permeabilidad del medio sea una función de salto, nuestro problema se denomina problema de Muskat no-homogéneo, Figura 3.

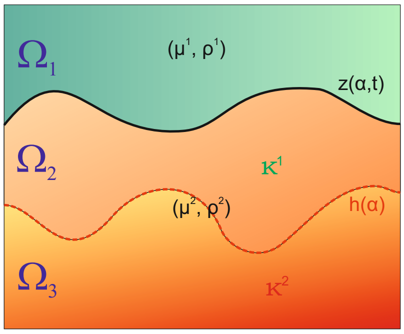


Figure 3: Problema de Muskat no-homogéneo

En este problema nos vamos a encontrar un medio dividido en dos regiones debido a los distintos valores de la permeabilidad

$$\kappa(x_1,x_2):=\begin{cases}\kappa^1 & x\in\Omega_1(t)\cup\Omega_2(t)=\mathbb{R}^2-\Omega_3,\\ \kappa^2 & x\in\Omega_3.\end{cases}$$

Por lo tanto, la influencia de esta permeabilidad tiene que ser tomada en cuenta. De este modo, las ecuaciones que describen nuestro sistema son ligeramente distintas.

De nuevo vamos a parametrizar la interfase entre nuestros fluidos por

$$z(\alpha,t)=\{(z_1(\alpha,t),z_2(\alpha,t)):\alpha\in\mathbb{R}\}$$

y

$$h(\alpha)=\{(h_1(\alpha),h_2(\alpha)):\alpha\in\mathbb{R}\}$$

para la curva, fijada en el tiempo, que separa las regiones con diferentes permeabilidades. Partiremos del caso en el que inicialmente estas curvas no se tocan.

Fijándonos en la ley de Darcy, podemos comprobar que la vorticidad vuelve a ser nula en el interior de las distintas regiones y que en el sentido de las distribuciones:

$$\omega(x,t) = \varpi_1(\alpha,t)\delta(x - z(\alpha,t)) + \varpi_2(\alpha,t)\delta(x - h(\alpha)),$$

donde ϖ_1 y ϖ_2 son las amplitudes de la vorticidad concentradas en z y h, respectivamente, véase [1]. Por la ley de Biot-Savart, procediendo del mismo modo que en el caso homogéneo:

$$\begin{aligned} u(x,t) &= \frac{1}{2\pi} PV \int_{\mathbb{R}} \frac{(x - z(\beta,t))^{\perp}}{|x - z(\beta,t)|^2} \varpi_1(\beta,t) d\beta + \frac{1}{2\pi} PV \int_{\mathbb{R}} \frac{(x - h(\beta))^{\perp}}{|x - h(\beta)|^2} \varpi_2(\beta,t) d\beta \\ &\equiv BR(\varpi_1, z)_x + BR(\varpi_2, h)_x. \end{aligned} \tag{6}$$

Ahora si calculamos los límites en la dirección normal a $z(\alpha,t)$ y $h(\alpha)$, obtenemos:

$$\begin{aligned} u^{\pm}(z(\alpha,t),t) &= BR(\varpi_1, z)_z(\alpha,t) + BR(\varpi_2, h)_z(\alpha,t) \mp \frac{1}{2} \frac{\varpi_1(\alpha,t)}{|\partial_\alpha z(\alpha,t)|^2} \partial_\alpha z(\alpha,t), \\ u^{\pm}(h(\alpha),t) &= BR(\varpi_1, z)_h(\alpha,t) + BR(\varpi_2, h)_h(\alpha,t) \mp \frac{1}{2} \frac{\varpi_2(\alpha,t)}{|\partial_\alpha h(\alpha)|^2} \partial_\alpha h(\alpha). \end{aligned}$$

Ya que $p^1(z(\alpha,t),t) = p^2(z(\alpha,t),t)$ tenemos,

$$\frac{\mu^2}{\kappa^1} u^-(z(\alpha,t),t) - \frac{\mu^1}{\kappa^1} u^+(z(\alpha,t),t) \cdot \partial_\alpha z(\alpha,t) = -g(\rho^2 - \rho^1) \partial_\alpha z_2(\alpha,t).$$

Usando los límites anteriores,

$$\begin{aligned} &\frac{\mu^2}{\kappa^1} u^-(z(\alpha,t),t) - \frac{\mu^1}{\kappa^1} u^+(z(\alpha,t),t) \cdot \partial_\alpha z(\alpha,t) = \\ &= \frac{\mu^2 - \mu^1}{\kappa^1} (BR(\varpi_1, z)_z + BR(\varpi_2, h)_z) \cdot \partial_\alpha z(\alpha,t) + \frac{\mu^2 + \mu^1}{2\kappa^1} \varpi_1(\alpha,t). \end{aligned}$$

Así,

$$\varpi_1(\alpha,t) = -2 \frac{\mu^2 - \mu^1}{\mu^2 + \mu^1} (BR(\varpi_1, z)_z + BR(\varpi_2, h)_z) \cdot \partial_\alpha z(\alpha,t) - 2\kappa^1 \frac{\rho^2 - \rho^1}{\mu^2 + \mu^1} g \partial_\alpha z_2(\alpha,t). \tag{7}$$

Para $\varpi_2(\alpha,t)$ procedemos de la misma forma. Ya que $p^2(h(\alpha),t) = p^3(h(\alpha),t)$ (recordemos que p^j es la presión en el dominio Ω^j) tenemos que,

$$\mu^2 \left(\frac{u^-(h(\alpha),t)}{\kappa^2} - \frac{u^+(h(\alpha),t)}{\kappa^1}\right) \cdot \partial_\alpha h(\alpha) = -\partial_\alpha (p^3(h(\alpha),t) - p^2(h(\alpha),t)) = 0$$

y que

$$\begin{aligned} &\mu^2 \left(\frac{u^-(h(\alpha),t)}{\kappa^2} - \frac{u^+(h(\alpha),t)}{\kappa^1}\right) \cdot \partial_\alpha h(\alpha) = \\ &= \mu^2 \left(\frac{1}{\kappa^2} - \frac{1}{\kappa^1}\right)(BR(\varpi_1, z)_h + BR(\varpi_2, h)_h) \cdot \partial_\alpha h(\alpha) + \frac{\mu^2}{2}\left(\frac{1}{\kappa^2} + \frac{1}{\kappa^1}\right) \varpi_2(\alpha,t). \end{aligned}$$

Por tanto,

$$\varpi_2(\alpha,t) = -2 \frac{\kappa^1 - \kappa^2}{\kappa^2 + \kappa^1} (BR(\varpi_1, z)_h + BR(\varpi_2, h)_h) \cdot \partial_\alpha h(\alpha). \tag{8}$$

Por último, añadimos términos tangenciales con el fin de obtener $|\partial_\alpha z(\alpha,t)|^2 \equiv A(t)$ y para ello tomamos,

$$
\begin{aligned}
c(\alpha,t)=&\frac{\alpha+\pi}{2\pi A(t)}\int_{\mathbb{T}}\partial_\alpha z(\beta,t)\cdot\partial_\alpha(BR(\varpi_1,z)_z+BR(\varpi_2,h)_z)d\beta\\
&-\int_{-\pi}^{\alpha}\frac{\partial_\alpha z(\beta,t)}{A(t)}\cdot\partial_\alpha(BR(\varpi_1,z)_z+BR(\varpi_2,h)_z)d\beta.
\end{aligned}
$$

Finalmente, el sistema que describe nuestro nuevo problema es:

$$
(IMP)\begin{cases}
z_t(\alpha,t)=BR(\varpi_1,z)_z(\alpha,t)+BR(\varpi_2,h)_z(\alpha,t)+c(\alpha,t)\partial_\alpha z(\alpha,t),\\
c(\alpha,t)=\frac{\alpha+\pi}{2\pi A(t)}\int_{\mathbb{T}}\partial_\alpha z(\beta,t)\cdot\partial_\alpha(BR(\varpi_1,z)_z+BR(\varpi_2,h)_z)d\beta\\
-\int_{-\pi}^{\alpha}\frac{\partial_\alpha z(\beta,t)}{A(t)}\cdot\partial_\alpha(BR(\varpi_1,z)_z+BR(\varpi_2,h)_z)d\beta,\\
\varpi_1(\alpha,t)=-2\frac{\mu^2-\mu^1}{\mu^2+\mu^1}(BR(\varpi_1,z)_z+BR(\varpi_2,h)_z)\cdot\partial_\alpha z(\alpha,t)-2\kappa^1\frac{\rho^2-\rho^1}{\mu^2+\mu^1}g\partial_\alpha z_2(\alpha,t),\\
\varpi_2(\alpha,t)=-2\frac{\kappa^1-\kappa^2}{\kappa^2+\kappa^1}(BR(\varpi_1,z)_h+BR(\varpi_2,h)_h)\cdot\partial_\alpha h(\alpha).
\end{cases}
$$

En este trabajo, nos vamos a hacer cargo del caso más genérico del problema de Muskat no-homogéneo. En el capítulo 2, probaremos existencia local en tiempo en espacios de Sobolev para el régimen estable. En este caso la condición de Rayleigh-Taylor, encargada de la estabilidad del problema, puede ser expresada de la forma:

$$
\sigma(\alpha,t)=\frac{\mu^2-\mu^1}{\kappa^1}(BR(\varpi_1,z)_z+BR(\varpi_2,h)_z)\cdot\partial_\alpha^{\perp}z(\alpha)+(\rho^2-\rho^1)g\partial_\alpha z_1(\alpha)>0.
$$

Algunos trabajos han descrito el caso en el que el dominio del fluido es $\mathbb{R}\times(-l,l)$ con $l>0$ [15, 18, 19] y han obtenido existencia local de soluciones clásicas en un régimen estable. Este caso con fronteras se puede entender como un problema de diferentes permeabilidades donde las permeabilidades exteriores se anulan. Para el caso con mismas viscosidades, es decir, $\mu^1=\mu^2$ la existencia local ha sido probada, en [1]. En este trabajo además consideran $h(\alpha)=(\alpha,-h_2)$, con $h_2>0$, entonces las ecuaciones de las amplitudes de la vorticidad son más sencillas:

$$
\begin{aligned}
\varpi_1(\alpha,t)&=-(\rho^2-\rho^1)\partial_\alpha z_2(\alpha,t),\\
\varpi_2(\alpha,t)&=\frac{\kappa^1-\kappa^2}{\kappa^1+\kappa^2}\frac{\kappa^1(\rho^2-\rho^1)}{\pi}PV\int_{\mathbb{R}}\frac{h_2+z_2(\alpha,t)}{|h(\alpha)-z(\beta)|^2}\partial_\alpha z_2(\beta,t)d\beta.
\end{aligned}
$$

En nuestro caso, con viscosidades diferentes, las expresiones (7) y (8) involucran las integrales de Birkhoff-Rott que dependen de ambas amplitudes, nos encontramos por tanto con un obstáculo delicado, necesitamos invertir un operador. El teorema principal de este capítulo es:

Theorem 0.0.2. *Sea $z_0(\alpha)\in H^k$, $h(\alpha)\in H^k$ para $k\geq 3$, $\mathcal{F}(z_0)(\alpha,\beta)\in L^\infty$, $\mathcal{F}(h)(\alpha,\beta)\in L^\infty$ y la distancia entre z_0 y h es distinta de cero. Entonces, si la condición de Rayleigh-Taylor se satisface, existe una solución clásica del Problema de Muskat no-homogéneo (IMP), $z\in\mathcal{C}^1([0,T],H^k(\mathbb{T}))$ donde $T=T(z_0)$.*

Para probar este Teorema 0.0.2 usaremos estimaciones de energía en las cuales es necesario controlar la evolución en tiempo de z, es decir, $\frac{d}{dt}\|z\|^2_{H^k}$ para $k\geq 3$. Una de las dificultades que encontraremos será el control de la norma H^k de $\varpi=(\varpi_1,\varpi_2)$, que exigirá el estudio del siguiente operador:

$$
\mathcal{T}(u_1,u_2)(\alpha)=\begin{pmatrix}T_1&T_2\\T_3&T_4\end{pmatrix}\begin{pmatrix}u_1\\u_2\end{pmatrix}
$$

donde

$$\begin{cases} T_1(u)(\alpha) = 2BR(u,z)_z(\alpha)\cdot\partial_\alpha z(\alpha),\\ T_2(u)(\alpha) = 2BR(u,h)_z(\alpha)\cdot\partial_\alpha z(\alpha),\\ T_3(u)(\alpha) = 2BR(u,z)_h(\alpha)\cdot\partial_\alpha h(\alpha),\\ T_4(u)(\alpha) = 2BR(u,h)_h(\alpha)\cdot\partial_\alpha h(\alpha). \end{cases}$$

El principal obstáculo será la estimación de la norma $H^{\frac{1}{2}}$ del operador inverso $(I+M\mathcal{T})^{-1}$ cuando $M = \begin{pmatrix} m_1 & 0\\ 0 & m_2\end{pmatrix}$ con $|m_i|\le 1$ para $i=1,2$. Para ello necesitaremos encontrar los operadores adecuados que nos permitan estimar la norma L^2 del operador $(I\pm\mathcal{T})^{-1}$. Todas estas cuestiones aparecen en el capítulo 2.1. Una vez superado este problema, estimamos $\varpi=(\varpi_1,\varpi_2)$ en la sección 2.2, las integrales de Birkhoff-Rott (6) en 2.3 y calcularemos las estimaciones a priori de nuestra frontera libre en 2.4-2.5.2. La evolución de la distancia entre z y h, de la condición cuerda-arco y del mínimo de la condición Rayleigh-Taylor también deben ser controladas (secciones 2.6, 2.7 y 2.8). Finalmente, terminaremos en la sección 2.9 uniendo todos los cálculos anteriores aplicando el método clásico de regularización del problema y conclusión de la existencia local buscada. Este trabajo ha sido publicado en Nonlinearity, [28].

Teniendo garantizada la existencia local del problema de Muskat no-homogéneo, el siguiente paso natural a dar es realizar el estudio de singularidades a tiempo finito. En el capítulo 3, de nuevo consideramos $\mu^1=\rho^1=0$ de la misma forma que ocurre con el caso homogéneo y probaremos:

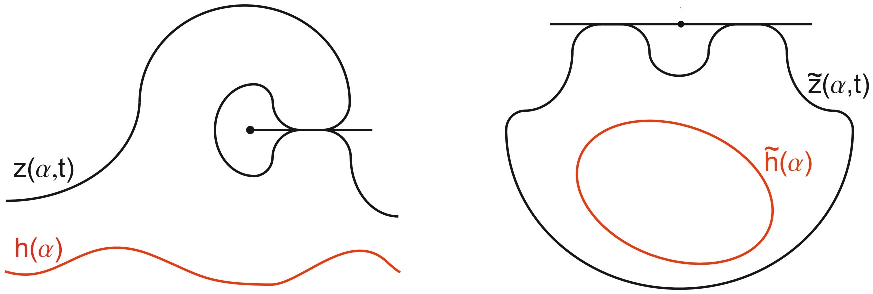
* No existencia de singularidades de tipo "splat", sección 3.1.

* Existencia de singularidades de tipo "splash", sección 3.2.

La idea de la demostración de la ausencia de singularidades de tipo "splat" es exactamente la misma que en el caso homogéneo, con las dificultades técnicas que conllevan las ecuaciones del caso no-homogéneo. El teorema principal de esta sección es:

Theorem 0.0.3. *Sea $z_0(\alpha)\in H^k$ y $h(\alpha)\in H^k$ para $k\ge 4$; $\sigma_0(\alpha)>0$, $\mathcal{F}(z_0)(\alpha,\beta)\in L^\infty$, $\mathcal{F}(h)\in L^\infty$ y la distancia entre z_0 y h es distinta de cero. Entonces el problema de Muskat no-homogéneo unifásico no colapsará en una singularidad de tipo "splat", es decir, no hay un tiempo finito donde existan intervalos disjuntos $I_1,I_2\in\mathbb{R}$ tales que $z(I_1,t)=z(I_2,t)$.*

En la subsección 3.1.1 presentamos las estimaciones a priori que nos proporcionan la analiticidad instantánea de la curva z cuando esta, inicialmente, satisface las condiciones cuerda-arco y Rayleigh-Taylor. La subsección 3.1.2 está dedicada a probar que la región de analiticidad no colapsa mientras la curva se mantenga suave y la condición cuerda-arco esté acotada. Finalizando con la subsección 3.1.3 donde se estudia el escenario adecuado usando la ya mencionada aplicación conforme P. Recordemos que esta transformación permitía separar los puntos de intersección de la interfase (véase Figura 4). Entonces consideramos através de esta transformación la evolución de la curva $\tilde{z}(\alpha,t)=P(z(\alpha,t))$. Una vez probada la analiticidad instantánea y el control de la región de analiticidad en el problema transformado, usando un argumento de contradicción finalizamos la demostración del teorema 0.0.3. Tanto en el estudio de la no existencia de singularidades de tipo "splat" como en el de existencia de singularidades "splash", necesitamos trabajar con el problema de Muskat no-homogéneo transformado mediante la aplicación conforme P. Las ecuaciones en este



(a) Dominio $\Omega(T)$ (b) Dominio $\tilde{\Omega}(T)$

Figure 4: Transformación del problema no-homogéneo mediante P

dominio transformado vendran dadas por:

$$\tilde{z}_t(\alpha,t) = Q^2(\alpha,t)(BR(\tilde{\varpi}_1,\tilde{z})_{\tilde{z}} + BR(\tilde{\varpi}_2,\tilde{h})_{\tilde{z}})(\alpha,t) + \tilde{c}(\alpha)\partial_\alpha \tilde{z}(\alpha,t)$$

donde

$$Q^2(\alpha,t) = |\frac{dP}{dw}(z(\alpha,t))|^2 = |\frac{dP}{dw}(P^{-1}(\tilde{z}(\alpha,t)))|^2,$$

$$\tilde{\varpi}_1(\alpha,t) = -2(BR(\tilde{\varpi}_1,\tilde{z})_{\tilde{z}} + BR(\tilde{\varpi}_2,\tilde{h})_{\tilde{z}})(\alpha,t)\cdot\partial_\alpha\tilde{z}(\alpha,t) - 2\frac{g\rho^2\kappa^1}{\mu^2}\partial_\alpha(P_2^{-1}(\tilde{z}(\alpha,t)))$$

$$\tilde{\varpi}_2(\alpha,t) = -2\frac{\kappa^2-\kappa^1}{\kappa^1+\kappa^2}(BR(\tilde{\varpi}_1,\tilde{z})_{\tilde{z}} + BR(\tilde{\varpi}_2,\tilde{h})_{\tilde{z}})(\alpha,t)\cdot\partial_\alpha\tilde{h}(\alpha,t)$$

y

$$\begin{aligned}\tilde{c}(\alpha,t) &= \frac{\alpha+\pi}{2\pi}\int_{\mathbb{T}}\frac{\partial_\beta\tilde{z}(\beta,t)}{|\partial_\beta\tilde{z}(\beta,t)|^2}\cdot\partial_\beta(Q^2(BR(\tilde{\varpi}_1,\tilde{z})_{\tilde{z}} + BR(\varpi_2,h)_z))(\beta,t)d\beta \\ &- \int_{-\pi}^{\alpha}\frac{\partial_\beta\tilde{z}(\beta,t)}{|\partial_\beta\tilde{z}(\beta,t)|^2}\cdot\partial_\beta(Q^2(BR(\varpi_1,z)_z + BR(\varpi_2,h)_z))(\beta,t)d\beta.\end{aligned}$$

Y la condición de Rayleigh-Taylor será:

$$\tilde{\sigma}(\alpha,t) = \frac{\mu^2}{\kappa^1}(BR(\tilde{\varpi}_1,\tilde{z})_{\tilde{z}} + BR(\tilde{\varpi}_2,\tilde{h})_{\tilde{z}})(\alpha,t)\cdot\partial_\alpha^\perp\tilde{z}(\alpha,t) + \rho^2 g\nabla P_2^{-1}(\tilde{z}(\alpha,t))\cdot\partial_\alpha^\perp\tilde{z}(\alpha,t) > 0.$$

La última parte, sección 3.2, está dedicada a la demostración del siguiente teorema:

Theorem 0.0.4. *Existe un conjunto abierto de curvas en H^3, que satisfacen las condiciones cuerda-arco y Rayleigh-Taylor, tales que para z_0 en este conjunto, la solución del problema de Muskat no-homogéneo unifásico viola la condición cuerda-arco en tiempo finito $T_s = T_s(z_0) > 0$. Además, esto ocurre de forma que $z(\alpha_1,T_s) = z(\alpha_2,T_s)$ con $\alpha_1 \neq \alpha_2$.*

Presentamos la demostración de este teorema dividiéndola en las siguientes subsecciones. En la subsección 3.2.1 describimos una familia de curvas z^l para las cuales hay una singularidad "splash" en un único punto x_s donde $x_s = z^l(\alpha_1) = z^l(\alpha_2)$ con $\alpha_1 \neq \alpha_2$ y $\partial_\alpha z_1^l(\alpha_1) = \partial_\alpha z_1^l(\alpha_2) = 0$. Usando la ley de Darcy y el lema de Hopf se obtiene que los puntos de intersección desaparecerán retrocediendo en el tiempo. Para continuar con la prueba es necesario transformar el problema de nuevo por nuestra aplicación P. Para el problema transformado, probamos existencia local como se puede observar en la subsección 3.2.2. En la subsección 3.2.3 se prueba un resultado de estabilidad para el problema transformado y en la subsección 3.2.4, concluimos la prueba del teorema 0.0.4 usando los teoremas de existencia y estabilidad.

Introduction - Abstract and conclusions

The Muskat problem is framed in the context of mathematical physics, in particular, in the study of partial differential equations that arise in fluids mechanics. This branch studies the movement of fluids as well as the forces that cause them, or the interactions between the fluid and the contour that limits them.

In this area it is imperative to mention the Navier-Stokes equations. They are the equations used to approximate the movement of usual fluids such as water, air or oil, and, thereof, they are a basic model in numerous sciences such as aeronautics, meteorology, hydraulics, etc.

The Navier-Stokes equations can be considered as the fundamental law, together with the laws of conservation of mass, that allows us to describe the movement of a fluid from certain initial and contour conditions.

The equations are:

$$\begin{cases} \rho(\frac{\partial u}{\partial t} + (u\cdot\nabla)u) = -\nabla p + \mu\Delta u + f, \\ \nabla\cdot u = 0, \\ \rho_t + (u\cdot\nabla)\rho = 0, \end{cases}$$

where $\nabla\cdot u = 0$ is the incompressible condition of the flux. However, these equations do not describe the dynamics of the fluids in a porous media. In this medium the fluid moves through voids (or pores) of a solid structure and the resistance offered by the solid structure must be taken into account. We will focus our study on this particular setting. There are many natural substances such as rocks, soils (eg: aquifers y oil sediments), zeolites, biological tissues (such as bones, wood and cork) or man-made materials such as cement and ceramics, which can be considered a porous media.

Fluids in porous media are of particular interest as they arise in a wide array of real problems coming from many areas of applied science and engineering: filtration, mechanics (acoustics, geomechanics, soil mechanics, rock mechanics), engineering (petroleum, Bio-remediation), geosciences (hydro-geology, geophysics), biology and biophysics, materials science, etc.

Given the many various applications of the behaviour of fluids in these settings and given that the Navier-Stokes equations do not provide a satisfactory model, as we have already pointed, the problem is how to proceed in an alternative way to determine this dynamics.

The work of Henry Darcy (1803-1858) provides a satisfactory answer to our needs: Darcy's Law. Darcy was an engineer of briges and roads, one of the people in charge of the design and construction of the water supply system of the city of Dijon. Approximately in 1850, Darcy discovered this experimental law that adequately describes the dynamics of the flow of an incompressible fluid in a porous medium:

$$\frac{\mu}{\kappa}u = -\nabla p - (0, g\rho),$$

where $(x,t) \in \mathbb{R}^2 \times \mathbb{R}^+$, $u = (u_1(x,t), u_2(x,t))$ is the incompressible velocity (i.e. $\nabla \cdot u = 0$), $p = p(x,t)$ is the pressure, $\mu = \mu(x,t)$ is the dynamic viscosity of the fluid, $\kappa = \kappa(x)$ is the permeability of the isotropic medium, $\rho = \rho(x,t)$ is the fluid density and g is the acceleration caused by gravity. With this new law our model for a fluid in a porous media is described by:

$$\begin{cases} \rho_t + (u \cdot \nabla)\rho = 0, \\ \frac{\mu}{\kappa}u = -\nabla p - (0, g\rho), \\ \nabla \cdot u = 0, \end{cases} \tag{9}$$

which allows us to study the central problem of this work, the *Muskat problem*.

Morris Muskat (1906-1998) was an American petroleum engineer who, in collaboration with Milan W. Meres, used Darcy's law to study the multi-phase flow of water, oil and gas in an oil field.

Specifically, the system (9) was studied in [27] where the evolution of the interface between two immiscible fluids of different nature in a porous medium is modeled.

This is a 2-dimensional free boundary problem where an interface is caused by the discontinuity between the viscosities and/or densities of the fluids:

$$(\mu, \rho)(x,t) := \begin{cases} (\mu^1, \rho^1) & x \in \Omega^1(t) \\ (\mu^2, \rho^2) & x \in \Omega^2(t) = \mathbb{R}^2 - \Omega^1(t) \end{cases}$$

where μ^1, ρ^1, μ^2 and ρ^2 are constants.

In this work we will start considering a porous medium in which the permeability remains constant, known as *homogeneous problem*.

For these conditions and considering a periodic medium in the horizontal variable, the evolution equations are obtained as follows:

Let the free boundary be parametrized by

$$\partial\Omega = \{z(\alpha,t) = (z_1(\alpha,t), z_2(\alpha,t)) : \alpha \in \mathbb{R}\}$$

so that the periodic condition

$$(z_1(\alpha + 2k\pi, t), z_2(\alpha + 2k\pi, t)) = (z_1(\alpha,t) + 2k\pi, z_2(\alpha,t))$$

holds with initial data $z(\alpha, 0) = z_0(\alpha)$. From Darcy's law, we deduce that the fluid is irrotational, i.e. $\omega = \nabla \times u = 0$, in the interior of each domain Ω^j for $j = 1, 2$ and thus, the vorticity is concentrated on the free boundary $z(\alpha,t)$ by a Dirac distribution as follows:

$$\omega(x,t) = \nabla^{\perp} \cdot u(x,t) = \varpi(\alpha,t)\delta(x - z(\alpha,t)) \tag{10}$$

where $\varpi(\alpha,t)$ represents the strength of the vorticity.

Due to incompressibility there exist a stream function $\psi(x,t)$ such that $u = \nabla^{\perp}\psi$. Since $\nabla \times \nabla^{\perp} = \Delta$, we obtain that $\omega = \Delta\psi$. Potential theory allows us to describe the solution of this

Poisson equation as

$$\psi(x,t)=\frac{1}{2\pi}\int_{\mathbb{R}^2}\ln|x-y|\omega(y,t)dy.$$

Then, using $\nabla^{\perp}=(-\partial_{x_2},\partial_{x_1})$ and (10) we have

$$u(x,t)=\frac{1}{2\pi}\int_{\mathbb{R}^2}\frac{(x-z(\beta,t))^{\perp}}{|x-z(\beta,t)|^2}\varpi(\beta,t)d\beta.$$

This equation is the analog of the well-known Biot–Savart law for the magnetic field induced by a current on a wire (c.f. [25]).

If we take limit of the velocity in the normal direction of boundary:

$$u^+(z(\alpha,t),t)=\lim_{\varepsilon\to 0^+}u(z(\alpha,t)+\varepsilon\partial_\alpha^{\perp}z(\alpha,t))=BR(z,\varpi)(\alpha,t)-\frac{\varpi(\alpha,t)}{2|\partial_\alpha z(\alpha,t)|^2}\partial_\alpha z(\alpha,t), \tag{11}$$

$$u^-(z(\alpha,t),t)=\lim_{\varepsilon\to 0^-}u(z(\alpha,t)+\varepsilon\partial_\alpha^{\perp}z(\alpha,t))=BR(z,\varpi)(\alpha,t)+\frac{\varpi(\alpha,t)}{2|\partial_\alpha z(\alpha,t)|^2}\partial_\alpha z(\alpha,t), \tag{12}$$

where $BR(z,\varpi)(\alpha,t)$ is known as Birkhoff-Rott integral:

$$BR(z,\varpi)(\alpha,t)=\frac{1}{2\pi}PV\int_{\mathbb{R}}\frac{(z(\alpha,t)-z(\beta,t))^{\perp}}{|z(\alpha,t)-z(\beta,t)|^2}\varpi(\beta,t)d\beta.$$

Since the free boundary moves with the fluid, $z(\alpha,t)$ will evolve with the velocity field $u(x,t)$:

$$z_t(\alpha,t)\cdot\partial_\alpha^{\perp}z(\alpha,t)=u^+(z(\alpha,t),t)\cdot\partial_\alpha^{\perp}z(\alpha,t)=u^-(z(\alpha,t),t)\cdot\partial_\alpha^{\perp}z(\alpha,t).$$

Therefore, it is sufficient to consider

$$z_t(\alpha,t)=BR(z,\varpi)(\alpha,t)+c(\alpha,t)\partial_\alpha z(\alpha,t)$$

for some function c. Let notice that different functions c correspond with different parametrizations of the curve. We will take $c(\alpha,t)$:

$$\begin{aligned}c(\alpha,t)=&\frac{\alpha+\pi}{2\pi}\int_{\mathbb{T}}\frac{\partial_\beta z(\beta,t)}{|\partial_\beta z(\beta,t)|^2}\cdot\partial_\beta BR(z,\varpi)(\beta,t)d\beta\\&-\int_{-\pi}^{\alpha}\frac{\partial_\beta z(\beta,t)}{|\partial_\beta z(\beta,t)|^2}\cdot\partial_\beta BR(z,\varpi)(\beta,t)d\beta.\end{aligned}$$

This choice allows us to remove the dependence on α from the modulous of the tangent vector $\partial_\alpha z(\alpha,t)$ (for more details see [11]), i.e.,

$$|\partial_\alpha z(\alpha,t)|^2=A(t).$$

We can close the system using Darcy's law and taking the dot product with $\partial_\alpha z(\alpha,t)$, we get:

$$\varpi(\alpha,t)=-2\frac{\mu^1-\mu^2}{\mu^1+\mu^2}BR(z,\varpi)(\alpha,t)\cdot\partial_\alpha z(\alpha,t)-2\kappa g\frac{\rho^2-\rho^1}{\mu^1+\mu^2}\partial_\alpha z_2(\alpha,t). \tag{13}$$

Using $p^1(z(\alpha,t))=p^2(z(\alpha,t))$, where p^j is the pressure in Ω^j, see [11], then:

$$(\mu^2u^--\mu^1u^+)\cdot\partial_\alpha z=-(\nabla p^2-\nabla p^1)\cdot\partial_\alpha z-g(\rho^2-\rho^1)\partial_\alpha z_2$$

$$= -\partial_n(p^2 - p^1) - g(\rho^2 - \rho^1)\partial_\alpha z_2 = -g(\rho^2 - \rho^1)\partial_\alpha z_2.$$

Using (11) and (12), we obtain (13)

Therefore, the interface of the Muskat problem is described by:

$$(P)\begin{cases} z_t(\alpha,t) = BR(z,\varpi)(\alpha,t) + c(\alpha,t)\partial_\alpha z(\alpha,t), \\ BR(z,\varpi)(\alpha,t) = \frac{1}{2\pi} PV\int \frac{(z(\alpha,t)-z(\beta,t))^\perp}{|z(\alpha,t)-z(\beta,t)|^2}\varpi(\beta,t)d\beta, \\ c(\alpha,t) = \frac{\alpha+\pi}{2\pi}\int_{\mathbb{T}} \frac{\partial_\beta z(\beta,t)}{|\partial_\beta z(\beta,t)|^2}\cdot\partial_\beta BR(z,\varpi)(\beta,t)d\beta \\ -\int_{-\pi}^{\alpha} \frac{\partial_\beta z(\beta,t)}{|\partial_\beta z(\beta,t)|^2}\cdot\partial_\beta BR(z,\varpi)(\beta,t)d\beta, \\ \varpi(\alpha,t) = -2\frac{\mu^1-\mu^2}{\mu^1+\mu^2}BR(z,\varpi)(\alpha,t)\cdot\partial_\alpha z(\alpha,t) - 2\kappa g\frac{\rho^2-\rho^1}{\mu^1+\mu^2}\partial_\alpha z_2(\alpha,t). \end{cases}$$

To proceed further is necessary to take into account two conditions of the problem: the first one is known as the arc-chord condition given by the function

$$\mathcal{F}(z)(\alpha,\beta,t) = \frac{\beta^2}{|z(\alpha)-z(\alpha-\beta)|^2}, \quad \alpha,\beta\in\mathbb{R}$$

with

$$\mathcal{F}(z)(\alpha,0,t) = \frac{1}{|\partial_\alpha z(\alpha,t)|^2}.$$

We will say that the arc-chord condition is satisfied when $\mathcal{F}(z)\in L^\infty$. This means that the parametrization is suitable and the interface do not present self-intersections.

Moreover, the well-known Rayleigh-Taylor condition determines the stability of the problem. Rayleigh [29] and Saffman-Taylor [30] gave a condition that must be satisfied for the linearized model in order to have a solution locally in time, namely that the normal component of the pressure gradient jump at the interface has to have a distinguished sign:

$$\sigma(\alpha,t) = -(\nabla p^2(z(\alpha,t)) - \nabla p^1(z(\alpha,t)))\cdot\partial_\alpha^\perp z(\alpha,t) > 0.$$

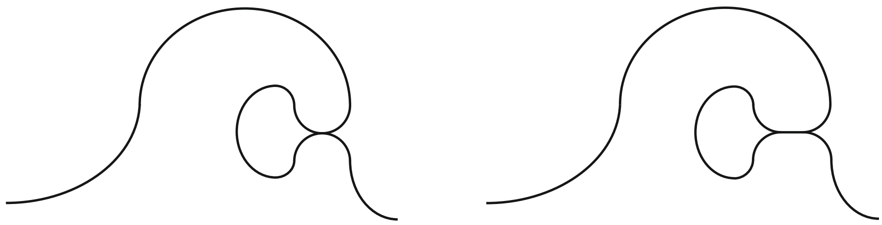
where p^j denote the pressure in Ω^j for $j=1,2$. This condition can be written as

$$\sigma(\alpha,t) = \frac{\mu^2-\mu^1}{\kappa}BR(z,\varpi)(\alpha,t)\cdot\partial_\alpha^\perp z(\alpha,t) + g(\rho^2-\rho^1)\partial_\alpha z_1(\alpha,t) > 0.$$

For more details consult [11]. Using Hopf's lemma, the Rayleigh-Taylor condition is satisfied for $\mu^1=\rho^1=0$, as it is demonstrated in section 1.2. For the case of equal viscosities ($\mu^1=\mu^2$), this condition holds when the more dense fluid lies below the interface [5]. This stability has been used to prove local existence in Sobolev spaces, when $\mu^1\neq\mu^2$ and $\rho^1\neq\rho^2\neq 0$, in [11].When $\mu^1=\mu^2$ and the interface is a graph, local existence are available in [13]. Taking the initial data on H^2 with $\mu^1=\rho^1=0$, local existence has been proved on [6]. For small data, the fact that $\sigma>0$ has been used to prove global existence as we can check in [8], [31], [19], [9], [7] and [23]. Furthermore, there exists initial data with $\sigma>0$ that in finite time turns to $\sigma<0$ (see [5] and [22]) and later in finite time the interface breaks down, [2].

This thesis is focus on the study of finite-time singularities. In particular, we will study the two types of finite time singularities shown for water waves in [4], the splash and splat singularities. These singularities are described for the scenario in which they consider the evolution of the free boundary of a region of water in the vacuum. For the study of this type of singularities in the Muskat problem, we will consider the *one-phase case*, that is, a single fluid in the vacuum with

$\mu^1 = \rho^1 = 0$.



(a) Splash-type singularity (b) Splat-type singularity

Figure 5: Finite time singularities

Roughly speaking, the splash-type singularity (Figure 5(a)) corresponds to the case where the fluid interface self-intersects at a single point (a rigorous definition can be found in [4]). This kind of singularity can be achieved for the one-phase Muskat problem see [3]. But it cannot be developed in the case in which $\mu^1 = \mu^2$ and $\rho^1 \neq \rho^2 \neq 0$, as we can see in [21].

The splat-type singularity (Figure 5(b)), is a variation of the splash in which the fluid interface self-intersects along an arc. This scenario has been shown to arise for the incompressible Euler equations in the water waves form, see [4], which considers the evolution of the free boundary of a water region in vacuum and irrotational velocity. In [16], these singularities have also been shown to exists for the case with vorticity. However, in this work will be proved the absence of splat singularities for the one-phase homogeneous Muskat problem (c.f. chapter 1). The main objective of chapter 1 is the demonstration of the following theorem:

Theorem 0.0.5. *Let $z_0(\alpha) \in H^k(\mathbb{T})$ for $k \geq 4$ and $\mathcal{F}(z_0)(\alpha,\beta) \in L^\infty$. Then the Muskat problem (P) will not break down in a splat singularity, i.e., there is no time where there exist disjoint intervals $I_1, I_2 \in \mathbb{R}$ such that $z(I_1,t) = z(I_2,t)$.*

For the study of this type of singularities we will consider the problem of one-phase Muskat, that is, $\mu^1 = \rho^1 = 0$. Therefore,

$$(\mu,\rho)(x,t) = \begin{cases} (0,0) & \text{for } x \in \mathbb{R}^2 - \Omega(t), \\ (\mu^2,\rho^2) & \text{for } x \in \Omega(t), \end{cases}$$

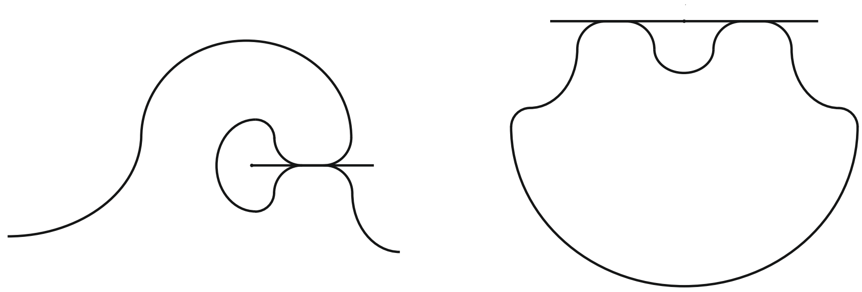
where $\Omega(t)$ is the domain which describes the fluid region.

The idea behind the proof is the following:

Suppose that a splat singularity is formed at a time T, that is, the interface $z(\alpha,t)$ collapses on itself in a curve at time $t = T$. If we start from a curve that is initially regular in our domain, namely $H^k(\mathbb{T})$ for $k \geq 4$, we will see that it becomes analytical instantaneously. Moreover, we will also have control over this analyticity as long as the regularity H^k of the curve and the arc-chord condition do not fail. But in our initial domain Ω, at time T, the arc-chord condition is not satisfied. Therefore we can not guarantee analyticity at that time. To solve this problem, we transform our domain using the conformal map P:

$$P(w) = (\tan(\frac{w}{2}))^{\frac{1}{2}}.$$

The branch of the root will be taken in such a way that it separates the self-intersecting points of the interface. This conformal map transforms our domain Ω in $\tilde{\Omega}$ as we can see in Figure 3.1, removing the splat singularity. The new contour evolution equation where we handle the splat



(a) $\Omega(T)$ domain (b) $\tilde{\Omega}(T)$ domian

Figure 6: Conformal map P

singularity is (see [3] for more details):

$$\tilde{z}_t(\alpha,t) = Q^2(\alpha,t)BR(\tilde{z},\tilde{\varpi})(\alpha,t) + \tilde{c}(\alpha)\partial_\alpha \tilde{z}(\alpha,t)$$

where

$$Q^2(\alpha,t) = |\frac{dP}{dw}(z(\alpha,t))|^2 = |\frac{dP}{dw}(P^{-1}(\tilde{z}(\alpha,t)))|^2,$$

$$\tilde{\varpi}(\alpha,t) = -2BR(\tilde{z},\tilde{\varpi})(\alpha,t)\cdot\partial_\alpha\tilde{z}(\alpha,t) - 2\frac{\rho^2}{\mu^2}\partial_\alpha(P_2^{-1}(\tilde{z}(\alpha,t)))$$

and

$$\tilde{c}(\alpha,t) = \frac{\alpha+\pi}{2\pi}\int_{\mathbb{T}}\frac{\partial_\beta\tilde{z}(\beta,t)}{|\partial_\beta\tilde{z}(\beta,t)|^2}\cdot\partial_\beta BR(\tilde{z},\tilde{\varpi})(\beta,t)d\beta$$
$$-\int_{-\pi}^{\alpha}\frac{\partial_\beta\tilde{z}(\beta,t)}{|\partial_\beta\tilde{z}(\beta,t)|^2}\cdot\partial_\beta BR(\tilde{z},\tilde{\varpi})(\beta,t)d\beta.$$

Finally we find the Rayleigh-Taylor condition in terms of $\tilde{z}$:

$$\tilde{\sigma}(\alpha,t) = \frac{\mu^2}{\kappa}BR(\tilde{z},\tilde{\varpi})(\alpha,t)\cdot\partial_\alpha^\perp\tilde{z}(\alpha,t) + \rho^2 g\nabla P_2^{-1}(\tilde{z}(\alpha,t))\cdot\partial_\alpha^\perp\tilde{z}(\alpha,t).$$

In this new domain, there exists a solution of the transformed problem $\tilde{z}(\alpha,t)$ defined for $0 < t \leq T$. We will prove that $\tilde{z}$ is analytic in $[0,T]$ since $\tilde{\sigma} > 0$ in $[0,T]$ and the arc-chord condition holds in $\tilde{\Omega}$; where $\tilde{\sigma}$ is the Rayleigh-Taylor condition in $\tilde{\Omega}$ and its sign do not change with respect to σ. Therefore, we have z analytic at $[0,T]$, in particular at T, thus we get a contradiction and the non-splat is proved.

The work done in this chapter has been published in a scientific paper in Transactions of the American Mathematical Society(see [14]).

Until now, our scenario has been a porous medium whose permeability remains constant. But we might wonder what happens if this varies?

For the case where the permeability of the medium is a step function, our problem is called the inhomogeneous Muskat problem, Figure 7.

In this problem we will find a medium divided in two regions due to the different values of

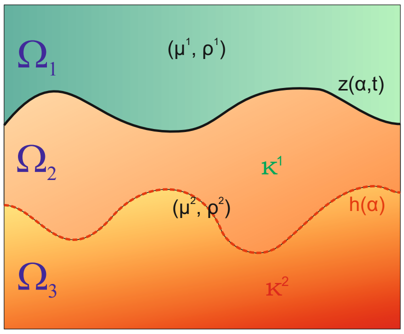


Figure 7: Inhomogeneous Muskat problem

permeability:

$$\kappa(x_1,x_2) := \begin{cases} \kappa^1 & x \in \Omega_1(t) \cup \Omega_2(t) = \mathbb{R}^2 - \Omega_3, \\ \kappa^2 & x \in \Omega_3. \end{cases}$$

Therefore, the influence of this permeability has to be taken into account. Thus, the equations describing our system are slightly different.

We parametrize the interface between two fluids by a curve

$$z(\alpha,t) = \{(z_1(\alpha,t), z_2(\alpha,t)) : \alpha \in \mathbb{R}\}$$

and

$$h(\alpha) = \{(h_1(\alpha), h_2(\alpha)) : \alpha \in \mathbb{R}\}$$

for the curve, fixed on time, which separates two different regions with different permeability (see Figure 7). We are going to consider that these two curves do not touch each other initially. Using Darcy's law we can see that the vorticity is nule inside the different regions and in distribution sense:

$$\omega(x,t) = \varpi_1(\alpha,t)\delta(x - z(\alpha,t)) + \varpi_2(\alpha,t)\delta(x - h(\alpha)),$$

where ϖ_1 and ϖ_2 are the strength of the vorticities concentrated on z and h, respectively, see [1]. Using the Biot-Savart law in the same way as in the homogeneous case, we know that

$$\begin{aligned} u(x,t) &= \frac{1}{2\pi} PV \int_{\mathbb{R}} \frac{(x - z(\beta,t))^{\perp}}{|x - z(\beta,t)|^2} \varpi_1(\beta,t) d\beta + \frac{1}{2\pi} PV \int_{\mathbb{R}} \frac{(x - h(\beta))^{\perp}}{|x - h(\beta)|^2} \varpi_2(\beta,t) d\beta \\ &\equiv BR(\varpi_1, z)_x + BR(\varpi_2, h)_x. \end{aligned} \tag{14}$$

If we calculate directional limits in the normal direction of $z(\alpha,t)$ and $h(\alpha)$:

$$u^{\pm}(z(\alpha,t),t) = BR(\varpi_1,z)_z(\alpha,t) + BR(\varpi_2,h)_z(\alpha,t) \mp \frac{1}{2} \frac{\varpi_1(\alpha,t)}{|\partial_\alpha z(\alpha,t)|^2} \partial_\alpha z(\alpha,t),$$

$$u^{\pm}(h(\alpha),t) = BR(\varpi_1,z)_h(\alpha,t) + BR(\varpi_2,h)_h(\alpha,t) \mp \frac{1}{2} \frac{\varpi_2(\alpha,t)}{|\partial_\alpha h(\alpha)|^2} \partial_\alpha h(\alpha).$$

Since $p^1(z(\alpha,t),t) = p^2(z(\alpha,t),t)$ we have,

$$\frac{\mu^2}{\kappa^1} u^+(z(\alpha,t),t) - \frac{\mu^1}{\kappa^1} u^-(z(\alpha,t),t) \cdot \partial_\alpha z(\alpha,t) = -g(\rho^2 - \rho^1)\partial_\alpha z_2(\alpha,t).$$

Using the above limits,

$$\begin{aligned}
&\frac{\mu^2}{\kappa^1}u^+(z(\alpha,t),t)-\frac{\mu^1}{\kappa^1}u^-(z(\alpha,t),t)\cdot\partial_\alpha z(\alpha,t)=\\
&=\frac{\mu^2-\mu^1}{\kappa^1}(BR(\varpi_1,z)_z+BR(\varpi_2,h)_z)\cdot\partial_\alpha z(\alpha,t)+\frac{\mu^2-\mu^1}{2\kappa^1}\varpi_1(\alpha,t).
\end{aligned}$$

Hence,

$$\varpi_1(\alpha,t)=-2\frac{\mu^2-\mu^1}{\mu^2+\mu^1}(BR(\varpi_1,z)_z+BR(\varpi_2,h)_z)\cdot\partial_\alpha z(\alpha,t)-2\kappa^1\frac{\rho^2-\rho^1}{\mu^2+\mu^1}g\partial_\alpha z_2(\alpha,t). \tag{15}$$

For $\varpi_2(\alpha,t)$ we proceed in the same way. Since $p^2(h(\alpha),t)=p^3(h(\alpha),t)$ then,

$$\mu^2\Big(\frac{u^-(h(\alpha),t)}{\kappa^2}-\frac{u^+(h(\alpha),t)}{\kappa^1}\Big)\cdot\partial_\alpha h(\alpha)=-\partial_\alpha(p^3(h(\alpha),t)-p^2(h(\alpha),t))=0$$

and

$$\begin{aligned}
&\mu^2\Big(\frac{u^-(h(\alpha),t)}{\kappa^2}-\frac{u^+(h(\alpha),t)}{\kappa^1}\Big)\cdot\partial_\alpha h(\alpha)=\\
&=\frac{\mu^2}{\kappa^2-\kappa^1}(BR(\varpi_1,z)_h+BR(\varpi_2,h)_h)\cdot\partial_\alpha h(\alpha)+\frac{\mu^2}{2(\kappa^2-\kappa^1)}\varpi_2(\alpha,t).
\end{aligned}$$

Therefore,

$$\varpi_2(\alpha,t)=-2\frac{\kappa^1-\kappa^2}{\kappa^2+\kappa^1}(BR(\varpi_1,z)_h+BR(\varpi_2,h)_h)\cdot\partial_\alpha h(\alpha). \tag{16}$$

We will add a tangential term in order to get $|\partial_\alpha z(\alpha,t)|^2\equiv A(t)$. For that we take,

$$\begin{aligned}
c(\alpha,t)=&\frac{\alpha+\pi}{2\pi A(t)}\int_{\mathbb{T}}\partial_\alpha z(\beta,t)\cdot\partial_\alpha(BR(\varpi_1,z)_z+BR(\varpi_2,h)_z)d\beta\\
&-\int_{-\pi}^{\alpha}\frac{\partial_\alpha z(\beta,t)}{A(t)}\cdot\partial_\alpha(BR(\varpi_1,z)_z+BR(\varpi_2,h)_z)d\beta.
\end{aligned}$$

Finally, the system which describes our problem is:

$$(IMP)\begin{cases}
z_t(\alpha,t)=BR(\varpi_1,z)_z(\alpha,t)+BR(\varpi_2,h)_z(\alpha,t)+c(\alpha,t)\partial_\alpha z(\alpha,t),\\
c(\alpha,t)=\frac{\alpha+\pi}{2\pi A(t)}\int_{\mathbb{T}}\partial_\alpha z(\beta,t)\cdot\partial_\alpha(BR(\varpi_1,z)_z+BR(\varpi_2,h)_z)d\beta\\
-\int_{-\pi}^{\alpha}\frac{\partial_\alpha z(\beta,t)}{A(t)}\cdot\partial_\alpha(BR(\varpi_1,z)_z+BR(\varpi_2,h)_z)d\beta,\\
\varpi_1(\alpha,t)=-2\frac{\mu^2-\mu^1}{\mu^2+\mu^1}(BR(\varpi_1,z)_z+BR(\varpi_2,h)_z)\cdot\partial_\alpha z(\alpha,t)-2\kappa^1\frac{\rho^2-\rho^1}{\mu^2+\mu^1}g\partial_\alpha z_2(\alpha,t),\\
\varpi_2(\alpha,t)=-2\frac{\kappa^1-\kappa^2}{\kappa^2+\kappa^1}(BR(\varpi_1,z)_h+BR(\varpi_2,h)_h)\cdot\partial_\alpha h(\alpha).
\end{cases}$$

In this thesis, we are going to take charge of the more general case of the inhomogenous Muskat problem. In the chapter 2, we will prove local existence in time in Sobolev spaces for stable regime. In this case Rayleigh-Taylor condition, responsible for the stability of the problem, can be expressed in the form:

$$\sigma(\alpha,t)=\frac{\mu^2-\mu^1}{\kappa^1}(BR(\varpi_1,z)_z+BR(\varpi_2,h)_z)\cdot\partial_\alpha^\perp z(\alpha)+(\rho^2-\rho^1)g\partial_\alpha z_1(\alpha)>0.$$

The case where the fluid domain is the strip $\mathbb{R}\times(-l,l)$ with $l>0$, has been studied in [15, 18, 19] and the authors obtain the existence of classical solution locally in time in the stable regime. This case with boundaries can be understood as a inhomogeneous problem where the outside permeabilities are zero. For the case with iquals viscosities, i.e., $\mu^1 = \mu^2$; local existence has been shown in [1]. In this scenario they consider $h(\alpha) = (\alpha, -h_2)$, with $h_2 > 0$, then the equations of the strength of the vorticities are simpler:

$$\begin{aligned}
\varpi_1(\alpha,t) &= -(\rho^2-\rho^1)\partial_\alpha z_2(\alpha,t),\\
\varpi_2(\alpha,t) &= \frac{\kappa^1-\kappa^2}{\kappa^1+\kappa^2}\frac{\kappa^1(\rho^2-\rho^1)}{\pi}PV\int_{\mathbb{R}}\frac{h_2+z_2(\alpha,t)}{|h(\alpha)-z(\beta)|^2}\partial_\alpha z_2(\beta,t)d\beta.
\end{aligned}$$

In our case, with different viscosities, the expressions (15) and (16) involves the Birkhoff-Rott integrals, so we find a delicate issue, we need to invert an operator. The main theorem of this chapter is:

Theorem 0.0.6. *Let $z_0(\alpha) \in H^k(\mathbb{T})$ for $k \geq 3$, $h(\alpha) \in H^k(\mathbb{T})$, $\mathcal{F}(z_0)(\alpha,\beta) \in L^\infty$, $\mathcal{F}(h)(\alpha,\beta) \in L^\infty$ and the distance between z_0 and h is different to zero. Then, if the Rayleigh-Taylor condition is satisfied, there exists a classical solution of the Muskat problem (IMP), $z \in \mathcal{C}^1([0,T], H^k(\mathbb{T}))$ where $T = T(z_0)$.*

In order to prove this theorem 0.0.6 we will use energy estimates, we will estimate the evolution on time of the interface z, i.e., $\frac{d}{dt}\|z\|^2_{H^k}$ for $k \geq 3$. As part of the technical calculations, we will need to estimate the norm H^k of $\varpi = (\varpi_1, \varpi_2)$, which will require the study of the following operator:

$$\mathcal{T}(u_1,u_2)(\alpha) = \begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

where

$$\begin{cases}
T_1(u)(\alpha) = 2BR(u,z)_z(\alpha)\cdot\partial_\alpha z(\alpha),\\
T_2(u)(\alpha) = 2BR(u,h)_z(\alpha)\cdot\partial_\alpha z(\alpha),\\
T_3(u)(\alpha) = 2BR(u,z)_h(\alpha)\cdot\partial_\alpha h(\alpha),\\
T_4(u)(\alpha) = 2BR(u,h)_h(\alpha)\cdot\partial_\alpha h(\alpha).
\end{cases}$$

In order to do this, we want to estimate the $H^{\frac{1}{2}}$-norm of the $(I+M\mathcal{T})^{-1}$ for $M = \begin{pmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{pmatrix}$ with $|\mu_i| \leq 1$ for $i = 1,2$. The main obstacle in this problem is to found some operators in order to estimate the L^2-norm of the inverse operator, see section 2.1.

Once we have the above estimates, we are qualified to estimate the strength of the vorticities $\varpi = (\varpi_1, \varpi_2)$ in section 2.2 and the Birkhoff-Rott integrals (14) in section 2.3.

For the purpose of study the local existence of classical solutions in Sobolev Spaces, we will use energy methods. For that we will need to obtain several a priori estimates for the curve $z(\alpha,t)$ with regularity H^k for $k \geq 3$. We present these computations in sections 2.4-2.5.2.

The other tools which we will need to show the estimates of the evolution of our energy, are the study of the evolution of the distance between z and h, the evolution of the arc-chord condition and the evolution of the minimum of the R-T condition. Sections 2.7, 2.6 and 2.8 are devoted to that.

Finally, after all these computations, in section 2.9 we follow the classical procedure and show the main theorem 0.0.6.

This work has led to the writing of a research article that has been published in Nonlinearity, [28].

Assuming the local existence of the inhomogeneous Muskat problem, the next natural step is to perform the study of finite-time singularities. In chapter 3, we consider again $\mu^1 = \rho^1 = 0$ as in the homogeneous case and we prove:

* Absence of splat singularities, Section 3.1

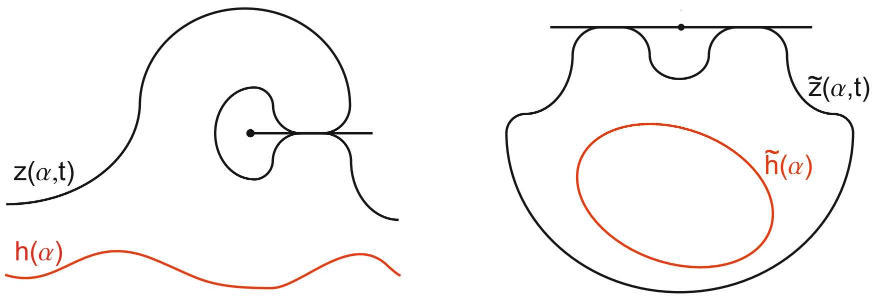
* Existence of the splash type singularities, Section 3.2

The idea behind the proof of the absence of splat singularities is exactly the same as in the homogeneous case, with some added technical difficulties coming from the equations of the inhomogeneous case. The main theorem of this section is:

Theorem 0.0.7. *Let $z_0(\alpha) \in H^k$ satisfying the Rayleigh-Taylor condition, $h(\alpha) \in H^k$ for $k \geq 4$, $\mathcal{F}(z_0)(\alpha, \beta) \in L^\infty$, $\mathcal{F}(h) \in L^\infty$ and the distance between z_0 and h is different to zero. Then the one-phase inhomogeneous Muskat problem will not break down in a splat singularity, i.e., there is no time where there exists disjoint intervals $I_1, I_2 \in \mathbb{R}$ such that $z(I_1, t) = z(I_2, t)$.*

In subsections 3.1.1 we present several a priori estimates that provide instant analyticity for a single curve that initially satisfies the arc-chord and Rayleigh-Taylor conditions. Subsection 3.1.2 is devoted to prove that the region of analyticity does not collapse as long as the curve remains smooth and the arc-chord condition remains bounded.

Finally in subsection 3.1.3, we explain how to treat the suitable scenario by using a conformal map P. Recall that this conformal map allow us to separate the self-intersecting points of the interface (see Figura 8). Then, we consider by this new transformation the evolution of the curve $\tilde{z}(\alpha, t) = P(z(\alpha, t))$. Once we prove instant analyticity and control of the strip of analyticity of the transformed problem, we can use a contradiction argument to finish the proof of theorem 0.0.7.



(a) $\Omega(T)$ domain (b) $\tilde{\Omega}(T)$ domain

Figure 8: Conformal map P

We need to work with the non-homogeneous Muskat problem transformed by conforming application, both in the study of non-existence of splat singularities and in the existence of splash singularities. The equations in this transformed domain will be given by:

$$\tilde{z}_t(\alpha, t) = Q^2(\alpha, t)(BR(\tilde{\varpi}_1, \tilde{z})_{\tilde{z}} + BR(\tilde{\varpi}_2, \tilde{h})_{\tilde{z}})(\alpha, t) + \tilde{c}(\alpha)\partial_\alpha \tilde{z}(\alpha, t)$$

where

$$Q^2(\alpha, t) = |\frac{dP}{dw}(z(\alpha, t))|^2 = |\frac{dP}{dw}(P^{-1}(\tilde{z}(\alpha, t)))|^2,$$

$$\tilde{\varpi}_1(\alpha, t) = -2(BR(\tilde{\varpi}_1, \tilde{z})_{\tilde{z}} + BR(\tilde{\varpi}_2, \tilde{h})_{\tilde{z}})(\alpha, t) \cdot \partial_\alpha \tilde{z}(\alpha, t) - 2\frac{g\rho^2\kappa^1}{\mu^2}\partial_\alpha(P_2^{-1}(\tilde{z}(\alpha, t)))$$

$$\tilde{\varpi}_2(\alpha,t) = -2\frac{\kappa^2-\kappa^1}{\kappa^1+\kappa^2}(BR(\tilde{\varpi}_1,\tilde{z})_{\tilde{z}} + BR(\tilde{\varpi}_2,\tilde{h})_{\tilde{z}})(\alpha,t)\cdot\partial_\alpha\tilde{h}(\alpha,t)$$

and

$$\begin{aligned}\tilde{c}(\alpha,t) &= \frac{\alpha+\pi}{2\pi}\int_{\mathbb{T}}\frac{\partial_\beta\tilde{z}(\beta,t)}{|\partial_\beta\tilde{z}(\beta,t)|^2}\cdot\partial_\beta(Q^2(BR(\tilde{\varpi}_1,\tilde{z})_{\tilde{z}} + BR(\varpi_2,h)_z))(\beta,t)d\beta\\ &-\int_{-\pi}^{\alpha}\frac{\partial_\beta\tilde{z}(\beta,t)}{|\partial_\beta\tilde{z}(\beta,t)|^2}\cdot\partial_\beta(Q^2(BR(\varpi_1,z)_z + BR(\varpi_2,h)_z))(\beta,t)d\beta.\end{aligned}$$

The Rayleigh-Taylor condition will be:

$$\tilde{\sigma}(\alpha,t) = \frac{\mu^2}{\kappa^1}(BR(\tilde{\varpi}_1,\tilde{z})_{\tilde{z}} + BR(\tilde{\varpi}_2,\tilde{h})_{\tilde{z}})(\alpha,t)\cdot\partial_\alpha^\perp\tilde{z}(\alpha,t) + \rho^2 g\nabla P_2^{-1}(\tilde{z}(\alpha,t))\cdot\partial_\alpha^\perp\tilde{z}(\alpha,t) > 0.$$

The last part, section 3.2, is dedicated to prove the existence of splash singularities at finite time. The main theorem in this section is:

Theorem 0.0.8. *There exists an open set of curves in H^3, satisfying the arc-chord and the Rayleigh-Taylor conditions, such that for any z_0 in this set, the solution of the one-phase inhomogeneous Muskat problem violates the arc-chord condition at a finite time $T_s = T_s(z_0) > 0$. Moreover, this holds in such a way that $z(\alpha_1,T_s) = z(\alpha_2,T_s)$ with $\alpha_1 \neq \alpha_2$.*

We show the proof of the above theorem by splitting the argument in the following subsections: In subsection 3.2.1 we describe a family of curves z^l for which there is splash singularity in the unique point x_s where $x_s = z^l(\alpha_1) = z^l(\alpha_2)$ with $\alpha_1 \neq \alpha_2$ and $\partial_\alpha z_1^l(\alpha_1) = \partial_\alpha z_1^l(\alpha_2) = 0$. Using Darcy's law and Hopf's lemma, we obtain that the self-intersection point is going to disappear going backward in time. In order to continue with the proof we need to transform our problem by using the conformal map P. For this transformed problem, we prove local existence as we can see in subsection 3.2.2. Subsection 3.2.3 is devoted to show a stability result for the transformed problem. Finally, in subsection 3.2.4 we conclude the proof of theorem 0.0.8 by using the existence and stability of the transformed problem.

Chapter 1

Non-splat singularities for the one-phase Muskat problem

For the water waves equations, the existence of splat singularities has been shown in [4], i.e., the interface self-intersects along an arc in finite time. The aim of this chapter is to show the absence of splat singularities for the incompressible fluid dynamics in porous media. For this problem we consider an homogeneous medium, $\kappa \equiv cte$ and one fluid in a dry region, $\mu^1 = \rho^1 = 0$. Therefore for the one-phase homogeneous Muskat problem, the evolution equations are:

$$(P)\begin{cases} z_t(\alpha,t) = BR(z,\varpi)(\alpha,t) + c(\alpha,t)\partial_\alpha z(\alpha,t) \\ BR(z,\varpi)(\alpha,t) = \frac{1}{2\pi}PV\int \frac{(z(\alpha,t)-z(\beta,t))^\perp}{|z(\alpha,t)-z(\beta,t)|^2}\varpi(\beta,t)d\beta \\ c(\alpha,t) = \frac{\alpha+\pi}{2\pi}\int_{\mathbb{T}} \frac{\partial_\beta z(\beta,t)}{|\partial_\beta z(\beta,t)|^2}\cdot\partial_\beta BR(z,\varpi)(\beta,t)d\beta \\ -\int_{-\pi}^{\alpha} \frac{\partial_\beta z(\beta,t)}{|\partial_\beta z(\beta,t)|^2}\cdot\partial_\beta BR(z,\varpi)(\beta,t)d\beta \\ \varpi(\alpha,t) = -2BR(z,\varpi)(\alpha,t)\cdot\partial_\alpha z(\alpha,t) - 2\kappa g\frac{\rho^2}{\mu^2}\partial_\alpha z_2(\alpha,t) \end{cases}$$

And the Rayleigh-Taylor condition can be written as:

$$\sigma(\alpha,t) = \frac{\mu^2}{\kappa}BR(z,\varpi)(\alpha,t)\cdot\partial_\alpha^\perp z(\alpha,t) + g\rho^2\partial_\alpha z_1(\alpha,t) > 0.$$

Remark 1.0.1. *Since we are studing the one-phase Muskat problem, the Rayleigh-Taylor condition holds and we do not need consider this condition as a hypothesis.*

The principal technique used here is Energy Methods which can be consulted in [26]. At first, sections are devoted to estimations on the Ω domain and to deal with the splat singularity we going to the $\tilde{\Omega}$ domain. The contour equations in the $\tilde{\Omega}$ domain are:

$$(\tilde{P})\begin{cases} \tilde{z}_t(\alpha,t) = Q^2(\alpha,t)BR(\tilde{z},\tilde{\varpi})(\alpha,t) + \tilde{c}(\alpha)\partial_\alpha\tilde{z}(\alpha,t) \\ Q^2(\alpha,t) = |\frac{dP}{dw}(z(\alpha,t))|^2 = |\frac{dP}{dw}(P^{-1}(\tilde{z}(\alpha,t)))|^2 \\ \tilde{\varpi}(\alpha,t) = -2BR(\tilde{z},\tilde{\varpi})(\alpha,t)\cdot\partial_\alpha\tilde{z}(\alpha,t) - 2\frac{\rho^2}{\mu^2}\partial_\alpha(P_2^{-1}(\tilde{z}(\alpha,t))) \\ \tilde{c}(\alpha,t) = \frac{\alpha+\pi}{2\pi}\int_{\mathbb{T}} \frac{\partial_\beta\tilde{z}(\beta,t)}{|\partial_\beta\tilde{z}(\beta,t)|^2}\cdot\partial_\beta BR(\tilde{z},\tilde{\varpi})(\beta,t)d\beta \\ -\int_{-\pi}^{\alpha} \frac{\partial_\beta\tilde{z}(\beta,t)}{|\partial_\beta\tilde{z}(\beta,t)|^2}\cdot\partial_\beta BR(\tilde{z},\tilde{\varpi})(\beta,t)d\beta \end{cases} \quad (1.1)$$

Of course we need to consider the Rayleigh-Taylor condition in terms of $\tilde{z}$

$$\tilde{\sigma}(\alpha,t)=\frac{\mu^2}{\kappa}BR(\tilde{z},\tilde{\varpi})(\alpha,t)\cdot\partial_\alpha^\perp\tilde{z}(\alpha,t)+\rho^2 g\nabla P_2^{-1}(\tilde{z}(\alpha,t))\cdot\partial_\alpha^\perp\tilde{z}(\alpha,t). \tag{1.2}$$

1.1 Estimates on $z(\alpha,t)$

Here we show the main estimates that provide instant analyticity into the strip $S(t)=\{\alpha+i\zeta : |\zeta|<\lambda t\}$ for each t. To do that we will need the following estimates from [11]:

$$\|\varpi\|_{H^k}\leq \exp C(\|\mathcal{F}(z)\|^2_{L^\infty}+\|z\|^2_{H^{k+1}}), \tag{1.3}$$

for $k\geq 2$.

$$\|BR(z,\varpi)\|_{H^k}\leq \exp C(\|\mathcal{F}(z)\|^2_{L^\infty}+\|z\|^2_{H^{k+1}}), \tag{1.4}$$

for $k\geq 2$. These estimates follows also into the complex strip S, since the time derivative plays no role and hence any extra term appears in relation with the terms in [11].

Remark 1.1.1. *Inequalities (1.3) and (1.4) can also be bounded by a polynomial function, see [12]. In our case, to prove instant analyticity and the decay of the strip, both estimates are valid.*

Let λ[1] be given in the definition of $L^2(S)$ and $H^k(S)$,

$$\|z\|^2_{L^2(S)}(t)=\sum_{\pm}\int_{\mathbb{T}}|z(\alpha\pm i\lambda t,t)-(\alpha\pm i\lambda t,0)|^2 d\alpha,$$

$$\|z\|^2_{H^k(S)}(t)=\|z\|^2_{L^2(S)}(t)+\sum_{\pm}\int_{\mathbb{T}}|\partial_\alpha^k z(\alpha\pm i\lambda t,t)|^2 d\alpha.$$

Remark 1.1.2. *Above $|\cdot|$ is the modulus of a vector in $\mathbb{C}^2$.*

1.1.1 Estimates for the $H^4(S)$ norm

We shall analyze the evolution of $\|z\|_{H^4(S)}(t)$. In order to simplify the exposition we write $z(\alpha,t)=z(\alpha)$ for a fixed t, and we denote $\alpha\pm i\lambda t\equiv\gamma$. It is easy to find that

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}}|z(\alpha\pm i\lambda t)-(\alpha\pm i\lambda t,0)|^2 d\alpha\leq \exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)}+\|z\|^2_{H^k(S)}), \tag{1.5}$$

for $k\geq 3$. Next, we check that

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}}|\partial_\alpha^4 z(\gamma)|^2 d\alpha=\sum_{j=1,2}\frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}}|\partial_\alpha^4 z_j(\gamma)|^2 d\alpha$$

where,

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}}|\partial_\alpha^4 z_j(\gamma)|^2 d\alpha=\Re\int_{\mathbb{T}}\overline{\partial_\alpha^4 z_j(\gamma)}(\partial_t(\partial_\alpha^4 z_j)(\gamma)\pm i\lambda\partial_\alpha^5 z_j(\gamma))d\alpha$$

then,

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}}|\partial_\alpha^4 z(\gamma)|^2 d\alpha=\Re\int_{\mathbb{T}}\overline{\partial_\alpha^4 z(\gamma)}\cdot\partial_\alpha^4 z_t(\gamma)d\alpha\pm\Re\int_{\mathbb{T}}\overline{\partial_\alpha^4 z(\gamma)}\cdot i\lambda\partial_\alpha^5 z(\gamma)d\alpha$$

[1] At the end of the proof of the Theorem 1.3, we can take any $\lambda<\frac{\min_\alpha(\sigma(\alpha,0))}{2}$

$\equiv I_1 + I_2$. Let us study I_2:

$$
\begin{aligned}
I_2 &= \Re \int_{\mathbb{T}} \overline{\partial_\alpha^4 z(\gamma)} \cdot \partial_\alpha^5 z(\gamma) i\lambda d\alpha = \int_{\mathbb{T}} \lambda(-\Re(\partial_\alpha^4 z)\Im(\partial_\alpha^5 z) + \Im(\partial_\alpha^4 z)\Re(\partial_\alpha^5 z))d\alpha \\
&= 2\lambda \int_{\mathbb{T}} \Im(\partial_\alpha^4 z)\Re(\partial_\alpha^5 z)d\alpha = -2\lambda \int_{\mathbb{T}} \Im(\partial_\alpha^4 z)\Re(\Lambda(H(\partial_\alpha^4 z)))d\alpha \\
&= -2\lambda \int_{\mathbb{T}} \Lambda^{\frac{1}{2}}(\Im(\partial_\alpha^4 z))\Re(\Lambda^{\frac{1}{2}} H(\partial_\alpha^4 z))d\alpha \leq 2\lambda \|\Lambda^{\frac{1}{2}}\Im(\partial_\alpha^4 z)\|_{L^2(S)} \|\Lambda^{\frac{1}{2}}\partial_\alpha^4 z\|_{L^2(S)} \\
&\leq 2\lambda \|\Lambda^{\frac{1}{2}}\partial_\alpha^4 z\|_{L^2(S)}^2,
\end{aligned}
$$

where Λ is defined by the Fourier transform $\widehat{\Lambda f}(\xi) = |\xi|\widehat{f}(\xi)$ and H is the Hilbert transform:

$$
\begin{aligned}
\Lambda(f)(x) &= \frac{1}{2\pi} PV \int \frac{f(x) - f(y)}{|x-y|^2} dy, \\
H(f)(x) &= \frac{1}{\pi} PV \int \frac{f(y)}{x-y} dy.
\end{aligned}
$$

Therefore,

$$
I_2 \leq 2\lambda \|\Lambda^{\frac{1}{2}}\partial_\alpha^4 z\|_{L^2(S)}^2.
$$

Since we have $z_t(\gamma) = BR(z,\varpi)(\gamma) + c(\gamma)\partial_\alpha z(\gamma)$, then:

$$
I_1 = \Re \int_{\mathbb{T}} \overline{\partial_\alpha^4 z(\gamma)} \cdot \partial_\alpha^4 BR(z,\varpi)(\gamma) d\alpha + \Re \int_{\mathbb{T}} \overline{\partial_\alpha^4 z(\gamma)} \cdot \partial_\alpha^4 (c(\gamma) \cdot \partial_\alpha z(\gamma)) d\alpha \equiv J_1 + J_2.
$$

We will estimate J_1 in the subsections 1.1.1.1 and 1.1.1.2 and J_2 in 1.1.1.3.

1.1.1.1 Integrable terms in $\partial_\alpha^4 BR(z,\varpi)$

We decompose $J_1 = I_3 + I_4 + I_5 + I_6 + I_7$, where:

$$
\begin{aligned}
I_3 &= \frac{1}{2\pi}\Re \int_{\mathbb{T}}\int_{\mathbb{R}} \overline{\partial_\alpha^4 z(\gamma)} \cdot \partial_\alpha^4 (\frac{(z(\gamma)-z(\gamma-\beta))^\perp}{|z(\gamma)-z(\gamma-\beta)|^2})\varpi(\gamma-\beta)d\beta d\alpha, \\
I_4 &= \frac{2}{\pi}\Re \int_{\mathbb{T}}\int_{\mathbb{R}} \overline{\partial_\alpha^4 z(\gamma)} \cdot \partial_\alpha^3 (\frac{(z(\gamma)-z(\gamma-\beta))^\perp}{|z(\gamma)-z(\gamma-\beta)|^2})\partial_\alpha\varpi(\gamma-\beta)d\beta d\alpha, \\
I_5 &= \frac{3}{\pi}\Re \int_{\mathbb{T}}\int_{\mathbb{R}} \overline{\partial_\alpha^4 z(\gamma)} \cdot \partial_\alpha^2 (\frac{(z(\gamma)-z(\gamma-\beta))^\perp}{|z(\gamma)-z(\gamma-\beta)|^2})\partial_\alpha^2\varpi(\gamma-\beta)d\beta d\alpha, \\
I_6 &= \frac{2}{\pi}\Re \int_{\mathbb{T}}\int_{\mathbb{R}} \overline{\partial_\alpha^4 z(\gamma)} \cdot \partial_\alpha (\frac{(z(\gamma)-z(\gamma-\beta))^\perp}{|z(\gamma)-z(\gamma-\beta)|^2})\partial_\alpha^3\varpi(\gamma-\beta)d\beta d\alpha, \\
I_7 &= \frac{1}{2\pi}\Re \int_{\mathbb{T}}\int_{\mathbb{R}} \overline{\partial_\alpha^4 z(\gamma)} \cdot \frac{(z(\gamma)-z(\gamma-\beta))^\perp}{|z(\gamma)-z(\gamma-\beta)|^2}\partial_\alpha^4\varpi(\gamma-\beta)d\beta d\alpha.
\end{aligned}
$$

Below we estimate the highest order term of each I_1. In order to estimate I_j for $j = 4, 5, 6$, we refer the reader to the paper [11]. We get,

$$
I_4 + I_5 + I_6 \leq \exp C(\|\mathcal{F}(z)\|_{L^\infty(S)}^2 + \|z\|_{H^4(S)}^2)
$$

The most singular terms for I_3 are those in which four derivatives appear. In order to simplify we write $\Delta\partial_\alpha^k z \equiv \partial_\alpha^k z(\gamma) - \partial_\alpha^k z(\gamma-\beta)$. One of the two singular terms of I_3 is given by

$$
K_1 = \frac{1}{2\pi}\Re \int_{\mathbb{T}}\int_{\mathbb{R}} \overline{\partial_\alpha^4 z(\gamma)} \cdot \frac{(\Delta\partial_\alpha^4 z)^\perp}{|\Delta z|^2}\varpi(\gamma-\beta)d\beta d\alpha,
$$

which we decompose in $K_1 = L_1 + L_2$, where

$$L_1 = \frac{1}{2\pi}\Re\int_{\mathbb{T}}\int_{\mathbb{R}} \overline{\partial_\alpha^4 z(\gamma)}\cdot(\Delta\partial_\alpha^4 z)^{\perp}\varpi(\gamma-\beta)\Big(\frac{1}{|z(\gamma)-z(\gamma-\beta)|^2}-\frac{1}{|\partial_\alpha z(\gamma)|^2\beta^2}\Big)d\beta d\alpha,$$
$$L_2 = \frac{1}{2\pi}\Re\int_{\mathbb{T}}\int_{\mathbb{R}} \overline{\partial_\alpha^4 z(\gamma)}\cdot(\Delta\partial_\alpha^4 z)^{\perp}\frac{\varpi(\gamma-\beta)}{|\partial_\alpha z(\gamma)|^2\beta^2}d\beta d\alpha.$$

Let us study L_1, if $\psi = \gamma-\beta+s\beta+t\beta-st\beta$, $\phi=\gamma-\beta+s\beta$ and

$$\begin{aligned}
B(\gamma,\beta) &\equiv \frac{1}{|z(\gamma)-z(\gamma-\beta)|^2}-\frac{1}{|\partial_\alpha z(\gamma)|^2\beta^2}\\
&= \frac{\beta\int_0^1\int_0^1\partial_\alpha^2 z(\psi)(1-s)dtds\cdot\int_0^1[\partial_\alpha z(\gamma)+\partial_\alpha z(\phi)]ds}{|z(\gamma)-z(\gamma-\beta)|^2|\partial_\alpha z(\gamma)|^2}\\
&= \frac{\beta\int_0^1\int_0^1\frac{\partial_\alpha^2 z(\psi)-\partial_\alpha^2 z(\gamma)}{|\psi-\gamma|^\delta}\beta^\delta(-1+s+t-st)^\delta(1-s)dtds\cdot\int_0^1[\partial_\alpha z(\gamma)+\partial_\alpha z(\phi)]ds}{|z(\gamma)-z(\gamma-\beta)|^2|\partial_\alpha z(\gamma)|^2}\\
&+\frac{\beta\partial_\alpha^2 z(\gamma)\cdot\int_0^1[\partial_\alpha z(\gamma)+\partial_\alpha z(\phi)]ds}{|z(\gamma)-z(\gamma-\beta)|^2|\partial_\alpha z(\gamma)|^2}\equiv B_1(\gamma,\beta)+B_2(\gamma,\beta)
\end{aligned}\tag{1.6}$$

we have,

$$\begin{aligned}
L_1 &= \frac{1}{2\pi}\Re\int_{\mathbb{T}}\int_{\mathbb{R}} \overline{\partial_\alpha^4 z(\gamma)}\cdot(\Delta\partial_\alpha^4 z)^{\perp}\varpi(\gamma-\beta)B_1(\gamma,\beta)d\beta d\alpha\\
&+\frac{1}{2\pi}\Re\int_{\mathbb{T}}\int_{\mathbb{R}} \overline{\partial_\alpha^4 z(\gamma)}\cdot(\Delta\partial_\alpha^4 z)^{\perp}\varpi(\gamma-\beta)B_2(\gamma,\delta)d\beta d\alpha\equiv M_1+M_2.
\end{aligned}$$

It is easy to check,

$$M_1\le C\|\mathcal{F}(z)\|^{\frac{3}{2}}_{L^\infty(S)}\|z\|_{\mathcal{C}^{2,\delta}(S)}\|\varpi\|_{L^\infty(S)}\|\partial_\alpha^4 z\|^2_{L^2(S)}.$$

Furthermore,

$$\begin{aligned}
B_2(\gamma,\beta) &= \frac{\beta^2\partial_\alpha^2 z(\gamma)\int_0^1\int_0^1\partial_\alpha^2 z(\eta)(s-1)dtds}{|z(\gamma)-z(\gamma-\beta)|^2|\partial_\alpha z(\gamma)|^2}\\
&+\frac{\beta\partial_\alpha^2 z(\gamma)2\partial_\alpha z(\gamma)}{|z(\gamma)-z(\gamma-\beta)|^2|\partial_\alpha z(\gamma)|^2}\equiv B_3(\gamma,\beta)+B_4(\gamma,\beta).
\end{aligned}$$

In the same way, we deal with M_2 and we have:

$$\begin{aligned}
M_2 &= \frac{1}{2\pi}\Re\int_{\mathbb{T}}\int_{\mathbb{R}} \overline{\partial_\alpha^4 z(\gamma)}\cdot(\Delta\partial_\alpha^4 z)^{\perp}\varpi(\gamma-\beta)B_3(\gamma,\beta)d\beta d\alpha\\
&+\frac{1}{2\pi}\Re\int_{\mathbb{T}}\int_{\mathbb{R}} \overline{\partial_\alpha^4 z(\gamma)}\cdot(\Delta\partial_\alpha^4 z)^{\perp}\varpi(\gamma-\beta)B_4(\gamma,\beta)d\beta d\alpha\equiv N_1+N_2.
\end{aligned}$$

It is clear that,

$$N_1\le C\|\mathcal{F}(z)\|^2_{L^\infty}\|z\|^2_{\mathcal{C}^2}\|\varpi\|_{L^\infty}\|\partial_\alpha^4 z\|^2_{L^2(S)}$$

and

$$\begin{aligned}
N_2 &= \frac{1}{2\pi}\Re\int_{\mathbb{T}}\int_{\mathbb{R}} \overline{\partial_\alpha^4 z(\gamma)}\cdot(\Delta\partial_\alpha^4 z)^{\perp}\varpi(\gamma-\beta)\frac{\beta\partial_\alpha^2 z(\gamma)2\partial_\alpha z(\gamma)}{|\partial_\alpha z(\gamma)|^2}B(\gamma,\beta)d\beta d\alpha\\
&+\frac{1}{2\pi}\Re\int_{\mathbb{T}}\int_{\mathbb{R}} \overline{\partial_\alpha^4 z(\gamma)}\cdot(\Delta\partial_\alpha^4 z)^{\perp}\varpi(\gamma-\beta)\frac{\partial_\alpha^2 z(\gamma)2\partial_\alpha z(\gamma)}{|\partial_\alpha z(\gamma)|^4\beta}d\beta d\alpha\equiv O_1+O_2.
\end{aligned}$$

Directly,

$$O_1 \leq C\|\mathcal{F}(z)\|^2_{L^\infty}\|z\|^2_{\mathcal{C}^2}\|z\|^2_{\mathcal{C}^1}\|\varpi\|_{L^\infty}\|\partial^4_\alpha z\|^2_{L^2(S)}$$

and we decompose,

$$\begin{aligned}
O_2 &= \frac{1}{2\pi}\Re\int_{\mathbb{T}}\int_{\mathbb{R}}\overline{\partial^4_\alpha z(\gamma)}\cdot(\partial^4_\alpha z(\gamma))^\perp\varpi(\gamma-\beta)\frac{\partial^2_\alpha z(\gamma)2\partial_\alpha z(\gamma)}{|\partial_\alpha z(\gamma)|^2\beta}d\beta d\alpha\\
&-\frac{1}{2\pi}\Re\int_{\mathbb{T}}\int_{\mathbb{R}}\overline{\partial^4_\alpha z(\gamma)}\cdot(\partial^4_\alpha z(\gamma-\beta))^\perp\varpi(\gamma-\beta)\frac{\partial^2_\alpha z(\gamma)2\partial_\alpha z(\gamma)}{|\partial_\alpha z(\gamma)|^2\beta}d\beta d\alpha\\
&\equiv P_1+P_2
\end{aligned}$$

where

$$\begin{aligned}
P_1 &= \frac{1}{2}\Re\int_{\mathbb{T}}\overline{\partial^4_\alpha z(\gamma)}\cdot(\partial^4_\alpha z(\gamma))^\perp\frac{\partial^2_\alpha z(\gamma)2\partial_\alpha z(\gamma)}{|\partial_\alpha z(\gamma)|^2}H(\varpi)d\alpha\\
&\leq C\|\mathcal{F}(z)\|^{\frac{1}{2}}_{L^\infty}\|z\|_{\mathcal{C}^2}\|\partial^4_\alpha z\|^2_{L^2(S)}\|\varpi\|_{\mathcal{C}^\delta}
\end{aligned}$$

and

$$\begin{aligned}
P_2 &= -\frac{1}{2\pi}\Re\int_{\mathbb{T}}\int_{\mathbb{R}}\overline{\partial^4_\alpha z(\gamma)}\cdot(\partial^4_\alpha z(\gamma-\beta))^\perp(\varpi(\gamma-\beta)-\varpi(\gamma))\frac{\partial^2_\alpha z(\gamma)2\partial_\alpha z(\gamma)}{|\partial_\alpha z(\gamma)|^2\beta}d\beta d\alpha\\
&-\frac{1}{2\pi}\Re\int_{\mathbb{T}}\int_{\mathbb{R}}\overline{\partial^4_\alpha z(\gamma)}\cdot(\partial^4_\alpha z(\gamma-\beta))^\perp\varpi(\gamma)\frac{\partial^2_\alpha z(\gamma)2\partial_\alpha z(\gamma)}{|\partial_\alpha z(\gamma)|^2\beta}d\beta d\alpha\\
&\leq C\|\mathcal{F}(z)\|^2_{L^\infty}\|z\|_{\mathcal{C}^2}\|z\|_{\mathcal{C}^1}\|\varpi\|_{\mathcal{C}^1}\|\partial^4_\alpha z\|^2_{L^2(S)}\\
&-\frac{1}{2}\Re\int_{\mathbb{T}}\overline{\partial^4_\alpha z(\gamma)}\cdot\varpi(\gamma)\frac{\partial^2_\alpha z(\gamma)2\partial_\alpha z(\gamma)}{|\partial_\alpha z(\gamma)|^2}H(\partial^4_\alpha z)d\alpha\\
&\leq C\|\mathcal{F}(z)\|^2_{L^\infty}\|z\|_{\mathcal{C}^2}\|z\|_{\mathcal{C}^1}\|\varpi\|_{\mathcal{C}^1}\|\partial^4_\alpha z\|^2_{L^2(S)}
\end{aligned}$$

Here we have used

$$\begin{aligned}
&\|H(f)\|_{L^p}\leq C\|f\|_{L^p}\quad\text{for } 1<p<\infty,\\
&\|H(f)\|_{L^\infty}\leq\|f\|_{\mathcal{C}^\delta}\quad\text{for } f\in\mathcal{C}^\delta\text{, and } 0<\delta<1.
\end{aligned}$$

Hence, using (1.3)

$$L_1\leq\exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)}+\|z\|^2_{H^4(S)}).$$

For L_2 we write $L_2=M_3+M_4$, with

$$\begin{aligned}
M_3 &= \frac{1}{2\pi}\Re\int_{\mathbb{T}}\int_{\mathbb{R}}\overline{\partial^4_\alpha z(\gamma)}\cdot(\Delta\partial^4_\alpha z)^\perp\frac{\varpi(\gamma-\beta)-\varpi(\gamma)}{|\partial_\alpha z(\gamma)|^2\beta^2}d\beta d\alpha,\\
M_4 &= \frac{1}{2\pi}\Re\int_{\mathbb{T}}\int_{\mathbb{R}}\overline{\partial^4_\alpha z(\gamma)}\cdot(\Delta\partial^4_\alpha z)^\perp\frac{\varpi(\gamma)}{|\partial_\alpha z(\gamma)|^2\beta^2}d\beta d\alpha.
\end{aligned}$$

Next we write

$$\begin{aligned}
M_3 &= \frac{1}{2\pi}\Re\int_{\mathbb{T}}\int_{\mathbb{R}}\overline{\partial^4_\alpha z(\gamma)}\cdot\partial^4_\alpha z(\gamma)\frac{\varpi(\gamma-\beta)-\varpi(\gamma)}{|\partial_\alpha z(\gamma)|^2\beta^2}d\beta d\alpha\\
&-\frac{1}{2\pi}\Re\int_{\mathbb{T}}\int_{\mathbb{R}}\overline{\partial^4_\alpha z(\gamma)}\cdot\partial^4_\alpha z(\gamma-\beta)\frac{\varpi(\gamma-\beta)-\varpi(\gamma)}{|\partial_\alpha z(\gamma)|^2\beta^2}d\beta d\alpha\equiv N_3+N_4
\end{aligned}$$

where

$$
\begin{aligned}
N_3 &= \Re\int_{\mathbb{T}} \overline{\partial_\alpha^4 z(\gamma)}\cdot\partial_\alpha^4 z(\gamma)\frac{\Lambda(\varpi)(\gamma)}{|\partial_\alpha z(\gamma)|^2}d\alpha\\
&\leq C\|\mathcal{F}(z)\|_{L^\infty(S)}\|\partial_\alpha^4 z\|^2_{L^2(S)}\|\Lambda\varpi\|_{L^\infty(S)} \leq C\|\mathcal{F}(z)\|_{L^\infty(S)}\|\partial_\alpha^4 z\|^2_{L^2(S)}\|\varpi\|_{\mathcal{C}^{1,\delta}(S)}
\end{aligned}
$$

and

$$
\begin{aligned}
N_4 &= -\frac{1}{2\pi}\Re\int_{\mathbb{T}}\int_{\mathbb{R}} \overline{\partial_\alpha^4 z(\gamma)}\cdot\partial_\alpha^4 z(\gamma-\beta)\frac{\int_0^1[\partial_\alpha\varpi(\gamma-s\beta)-\partial_\alpha\varpi(\gamma)]ds}{|\partial_\alpha z(\gamma)|^2\beta}d\beta d\alpha\\
&-\frac{1}{2\pi}\Re\int_{\mathbb{T}}\int_{\mathbb{R}} \overline{\partial_\alpha^4 z(\gamma)}\cdot\partial_\alpha^4 z(\gamma-\beta)\frac{\partial_\alpha\varpi(\gamma)}{|\partial_\alpha z(\gamma)|^2\beta}d\beta d\alpha\\
&\leq C\|\mathcal{F}(z)\|_{L^\infty(S)}\|\partial_\alpha^4 z\|^2_{L^2(S)}\|\varpi\|_{\mathcal{C}^2(S)} - \frac{1}{2}\Re\int_{\mathbb{T}} \overline{\partial_\alpha^4 z(\gamma)}\cdot H(\partial_\alpha^4 z)(\gamma)\frac{\partial_\alpha\varpi(\gamma)}{|\partial_\alpha z(\gamma)|^2}d\alpha\\
&\leq \exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)} + \|z\|^2_{H^4(S)}).
\end{aligned}
$$

For M_4,

$$
\begin{aligned}
M_4 &= \Re\int_{\mathbb{T}} \overline{\partial_\alpha^4 z(\gamma)}\cdot\Lambda(\partial_\alpha^4 z^\perp)(\gamma)\frac{\varpi(\gamma)}{|\partial_\alpha z(\gamma)|^2}d\alpha\\
&= \int_{\mathbb{T}}\Im(\frac{\varpi}{A(t)})(-\Re(\partial_\alpha^4 z)\cdot\Im(\Lambda(\partial_\alpha^4 z^\perp)) + \Im(\partial_\alpha^4 z)\cdot\Re(\Lambda(\partial_\alpha^4 z^\perp)))d\alpha\\
&+ \int_{\mathbb{T}}\Re(\frac{\varpi}{A(t)})(\Re(\partial_\alpha^4 z)\cdot\Re(\Lambda(\partial_\alpha^4 z^\perp)) + \Im(\partial_\alpha^4 z)\cdot\Im(\Lambda(\partial_\alpha^4 z^\perp)))d\alpha\\
&\equiv N_5 + N_6.
\end{aligned}
$$

Now we take,

$$
\begin{aligned}
N_6 &= \int_{\mathbb{T}}\Re(\frac{\varpi}{A(t)})(-\Re(\partial_\alpha^4 z_1)\Re(\Lambda(\partial_\alpha^4 z_2)) + \Re(\partial_\alpha^4 z_2)\Re(\Lambda(\partial_\alpha^4 z_1)))d\alpha\\
&+ \int_{\mathbb{T}}\Re(\frac{\varpi}{A(t)})(-\Im(\partial_\alpha^4 z_1)\Im(\Lambda(\partial_\alpha^4 z_2)) + \Im(\partial_\alpha^4 z_2)\Im(\Lambda(\partial_\alpha^4 z_1)))d\alpha\\
&\equiv O_3 + O_4
\end{aligned}
$$

where it is easy to find a commutator formula such that, using (see [24])

$$
\|\Lambda(fg) - g\Lambda(f)\|_{L^2} \leq \|g\|_{\mathcal{C}^{1,\delta}}\|f\|_{L^2}, \tag{1.7}
$$

we get

$$
\begin{aligned}
O_3 &= \int_{\mathbb{T}}(-\Lambda(\Re(\frac{\varpi}{A(t)})\Re(\partial_\alpha^4 z_1)) + \Re(\frac{\varpi}{A(t)})\Re(\Lambda(\partial_\alpha^4 z_1)))\Re(\partial_\alpha^4 z_2)d\alpha\\
&\leq C\|\Re(\frac{\varpi}{A(t)})\|_{\mathcal{C}^{1,\delta}}\|\partial_\alpha^4 z\|^2_{L^2(S)}.
\end{aligned}
$$

In the same way,

$$
O_4 \leq C\|\Re(\frac{\varpi}{A(t)})\|_{\mathcal{C}^{1,\delta}(S)}\|\partial_\alpha^4 z\|^2_{L^2(S)}.
$$

Thus,

$$N_6 \leq \exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)} + \|z\|^2_{H^4(S)}).$$

For N_5 we have,

$$\begin{aligned}
N_5 &= \int_{\mathbb{T}} \Im(\frac{\varpi}{A(t)})(\Re(\partial_\alpha^4 z^\perp)\cdot \Im(\Lambda(\partial_\alpha^4 z)) + \Im(\partial_\alpha^4 z)\cdot \Re(\Lambda(\partial_\alpha^4 z^\perp)))d\alpha \\
&= \int_{\mathbb{T}} (\Lambda(\Im(\frac{\varpi}{A(t)})\Re(\partial_\alpha^4 z^\perp)) - \Im(\frac{\varpi}{A(t)})\Re(\Lambda(\partial_\alpha^4 z^\perp)))\cdot \Im(\partial_\alpha^4 z)d\alpha \\
&+ 2\int_{\mathbb{T}} \Im(\frac{\varpi}{A(t)})\Re(\Lambda(\partial_\alpha^4 z^\perp))\cdot \Im(\partial_\alpha^4 z)d\alpha \\
&\equiv O_5 + O_6.
\end{aligned}$$

Then,

$$O_5 \leq C\|\Im(\frac{\varpi}{A(t)})\|_{C^{1,\delta}(S)}\|\partial_\alpha^4 z\|^2_{L^2(S)} \leq \exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)} + \|z\|^2_{H^4(S)})$$

and

$$\begin{aligned}
O_6 &= 2\int_{\mathbb{T}} \Lambda^{\frac{1}{2}}(\Im(\frac{\varpi}{A(t)})\Im(\partial_\alpha^4 z))\cdot \Re(\Lambda^{\frac{1}{2}}(\partial_\alpha^4 z^\perp))d\alpha \leq 2\|\Lambda^{\frac{1}{2}}(\Im(\frac{\varpi}{A(t)})\Im(\partial_\alpha^4 z))\|_{L^2(S)}\|\Lambda^{\frac{1}{2}}\partial_\alpha^4 z\|_{L^2(S)} \\
&\leq C\|\Im(\frac{\varpi}{A(t)})\|_{H^2(S)}(\|\partial_\alpha^4 z\|_{L^2(S)} + \|\Lambda^{\frac{1}{2}}\partial_\alpha^4 z\|_{L^2(S)})\|\Lambda^{\frac{1}{2}}\partial_\alpha^4 z\|_{L^2(S)} \\
&\leq C\|\Im(\frac{\varpi}{A(t)})\|_{H^2(S)}(\frac{\|\partial_\alpha^4 z\|^2_{L^2(S)}}{2} + \frac{\|\Lambda^{\frac{1}{2}}\partial_\alpha^4 z\|^2_{L^2(S)}}{2}) + C\|\Im(\frac{\varpi}{A(t)})\|_{H^2(S)}\|\Lambda^{\frac{1}{2}}\partial_\alpha^4 z\|^2_{L^2(S)} \\
&\leq \exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)} + \|z\|^2_{H^4(S)}) + C\|\Im(\frac{\varpi}{A(t)})\|_{H^2(S)}\|\Lambda^{\frac{1}{2}}\partial_\alpha^4 z\|^2_{L^2(S)}.
\end{aligned}$$

Concluding,

$$K_1 \leq \exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)} + \|z\|^2_{H^4(S)}) + C\|\Im(\frac{\varpi}{A(t)})\|_{H^2(S)}\|\Lambda^{\frac{1}{2}}\partial_\alpha^4 z\|^2_{L^2(S)}.$$

The other singular term with four derivatives inside I_3 is given by

$$K_2 = -\frac{1}{\pi}\Re\int_{\mathbb{T}}\int_{\mathbb{R}} \overline{\partial_\alpha^4 z(\gamma)}\cdot \frac{(\Delta z)^\perp}{|\Delta z|^4}(\Delta z\cdot \Delta\partial_\alpha^4 z)\varpi(\gamma-\beta)d\beta d\alpha.$$

Here we take $K_2 = L_3 + L_4 + L_5$ where

$$\begin{aligned}
L_3 &= -\frac{1}{\pi}\Re\int_{\mathbb{T}}\int_{\mathbb{R}} \overline{\partial_\alpha^4 z(\gamma)}\cdot (\frac{(\Delta z)^\perp}{|\Delta z|^4} - \frac{\partial_\alpha z(\gamma)^\perp}{|\partial_\alpha z(\gamma)|^4\beta^3})(\Delta z - \beta\partial_\alpha z(\gamma))\cdot \varpi(\gamma-\beta)\Delta\partial_\alpha^4 z d\beta d\alpha, \\
L_4 &= -\frac{1}{\pi}\Re\int_{\mathbb{T}}\int_{\mathbb{R}} \overline{\partial_\alpha^4 z(\gamma)}\cdot (\frac{(\Delta z)^\perp}{|\Delta z|^4} - \frac{\partial_\alpha z(\gamma)^\perp}{|\partial_\alpha z(\gamma)|^4\beta^3})(\beta\partial_\alpha z(\gamma)\cdot \Delta\partial_\alpha^4 z)\varpi(\gamma-\beta)d\beta d\alpha, \\
L_5 &= -\frac{1}{\pi}\Re\int_{\mathbb{T}}\int_{\mathbb{R}} \overline{\partial_\alpha^4 z(\gamma)}\cdot \frac{\partial_\alpha z(\gamma)^\perp}{|\partial_\alpha z(\gamma)|^4\beta^3}(\Delta z\cdot \Delta\partial_\alpha^4 z)\varpi(\gamma-\beta)d\beta d\alpha.
\end{aligned}$$

We compute

$$C(\gamma,\beta) = \frac{(\Delta z)^\perp}{|\Delta z|^4} - \frac{\partial_\alpha z(\gamma)^\perp}{|\partial_\alpha z(\gamma)|^4\beta^3} = \frac{\beta^2\int_0^1\int_0^1 \partial_\alpha^2 z^\perp(\eta)(s-1)dsdt}{|\Delta z|^4}$$

$$+\frac{\beta^2\partial_\alpha z^\perp(\gamma)\int_0^1\int_0^1\partial_\alpha^2 z(\psi)(1-s)dsdt\cdot\int_0^1[\partial_\alpha z(\gamma)+\partial_\alpha z(\phi)]ds\int_0^1[|\partial_\alpha z(\gamma)|^2+|\partial_\alpha z(\phi)|^2]ds}{|\Delta z|^4|\partial_\alpha z(\gamma)|^4} \tag{1.8}$$
$$\equiv C_1(\gamma,\beta)+C_2(\gamma,\beta)$$

and

$$\Delta z-\beta\partial_\alpha z(\gamma)=\beta^2\int_0^1\int_0^1\partial_\alpha^2 z(\eta)(s-1)dtds$$

where $\eta=\gamma-t\beta+st\beta$, allowing us to obtain the desired estimate for the term L_3.

Next we split $L_4=M_5+M_6$ since $\Delta\partial_\alpha^4 z=\partial_\alpha^4 z(\gamma)-\partial_\alpha^4 z(\gamma-\beta)$:

$$M_5=-\frac{1}{\pi}\Re\int_{\mathbb{T}}\int_{\mathbb{R}}\overline{\partial_\alpha^4 z(\gamma)}\cdot C(\gamma,\beta)(\beta\partial_\alpha z(\gamma)\cdot\partial_\alpha^4 z(\gamma))\varpi(\gamma-\beta)d\beta d\alpha,$$
$$M_6=\frac{1}{\pi}\Re\int_{\mathbb{T}}\int_{\mathbb{R}}\overline{\partial_\alpha^4 z(\gamma)}\cdot C(\gamma,\beta)(\beta\partial_\alpha z(\gamma)\cdot\partial_\alpha^4 z(\gamma-\beta))\varpi(\gamma-\beta)d\beta d\alpha.$$

By following the same approach for L_1 we have,

$$|M_5|\le C\|\mathcal{F}(z)\|^2_{L^\infty(S)}\|z\|^2_{\mathcal{C}^2(S)}\|\varpi\|_{L^\infty(S)}\|\partial_\alpha^4 z\|^2_{L^2(S)}$$
$$+|\Re\int_{\mathbb{T}}\overline{\partial_\alpha^4 z(\gamma)}\cdot\partial_\alpha^2 z^\perp(\gamma)(\partial_\alpha z(\gamma)\cdot\partial_\alpha^4 z(\gamma))\frac{H(\varpi)(\gamma)}{|\partial_\alpha z(\gamma)|^4}d\alpha|$$

and

$$|M_6|\le C\|\mathcal{F}(z)\|^2_{L^\infty(S)}\|z\|^2_{\mathcal{C}^2(S)}\|\varpi\|_{L^\infty(S)}\|\partial_\alpha^4 z\|^2_{L^2(S)}$$
$$+|\Re\int_{\mathbb{T}}\overline{\partial_\alpha^4 z(\gamma)}\cdot\partial_\alpha^2 z^\perp(\gamma)(\partial_\alpha z(\gamma)\cdot H(\partial_\alpha^4 z)(\gamma))\frac{\varpi(\gamma)}{|\partial_\alpha z(\gamma)|^4}d\alpha|.$$

Then, the term L_4 is controlled. To conclude the estimates of K_2, we need to see what happens with the term L_5

$$L_5=-\frac{1}{\pi}\Re\int_{\mathbb{T}}\int_{\mathbb{R}}\overline{\partial_\alpha^4 z(\gamma)}\cdot\frac{\partial_\alpha^\perp z(\gamma)}{|\partial_\alpha z(\gamma)|^4}(\int_0^1[\partial_\alpha z(\phi)-\partial_\alpha z(\gamma)]ds\cdot\Delta\partial_\alpha^4 z)\frac{\varpi(\gamma-\beta)}{\beta^2}d\beta d\alpha$$
$$-\frac{1}{\pi}\Re\int_{\mathbb{T}}\int_{\mathbb{R}}\overline{\partial_\alpha^4 z(\gamma)}\cdot\frac{\partial_\alpha^\perp z(\gamma)}{|\partial_\alpha z(\gamma)|^4}(\partial_\alpha z(\gamma)\cdot\Delta\partial_\alpha^4 z)\frac{\varpi(\gamma-\beta)}{\beta^2}d\beta d\alpha$$
$$\equiv M_7+M_8.$$

For M_7 we proceed in the same way as in L_4 and we get:

$$M_7\le\exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)}+\|z\|^2_{H^4(S)})$$
$$-\Re\int_{\mathbb{T}}\overline{\partial_\alpha^4 z(\gamma)}\cdot\frac{\partial_\alpha^\perp z(\gamma)}{|\partial_\alpha z(\gamma)|^4}(\partial_\alpha^2 z(\gamma)\cdot\partial_\alpha^4 z(\gamma))H(\varpi)(\gamma)d\alpha$$
$$+\Re\int_{\mathbb{T}}\overline{\partial_\alpha^4 z(\gamma)}\cdot\frac{\partial_\alpha^\perp z(\gamma)}{|\partial_\alpha z(\gamma)|^4}(\partial_\alpha^2 z(\gamma)\cdot H(\partial_\alpha^4 z)(\gamma))\varpi(\gamma)d\alpha.$$

Then,

$$M_7\le\exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)}+\|z\|^2_{H^4(S)}).$$

To control the term M_8, we decompose it as follows,

$$
\begin{aligned}
M_8 = & -\frac{1}{\pi}\Re\int_{\mathbb{T}}\int_{\mathbb{R}}\overline{\partial_\alpha^4 z(\gamma)}\cdot\frac{\partial_\alpha^\perp z(\gamma)}{|\partial_\alpha z(\gamma)|^4}(\partial_\alpha z(\gamma)\cdot\Delta\partial_\alpha^4 z)\frac{\varpi(\gamma-\beta)-\varpi(\gamma)}{\beta^2}d\beta d\alpha \\
& -\frac{1}{\pi}\Re\int_{\mathbb{T}}\int_{\mathbb{R}}\overline{\partial_\alpha^4 z(\gamma)}\cdot\frac{\partial_\alpha^\perp z(\gamma)}{|\partial_\alpha z(\gamma)|^4}(\partial_\alpha z(\gamma)\cdot\Delta\partial_\alpha^4 z)\frac{\varpi(\gamma)}{\beta^2}d\beta d\alpha \\
& \equiv N_7+N_8.
\end{aligned}
$$

Since $\Delta\partial_\alpha^4 z=\partial_\alpha^4 z(\gamma)-\partial_\alpha^4 z(\gamma-\beta)$ we have,

$$
\begin{aligned}
N_7 = & -\frac{1}{\pi}\Re\int_{\mathbb{T}}\int_{\mathbb{R}}\overline{\partial_\alpha^4 z(\gamma)}\cdot\frac{\partial_\alpha^\perp z(\gamma)}{|\partial_\alpha z(\gamma)|^4}(\partial_\alpha z(\gamma)\cdot\partial_\alpha^4 z(\gamma))\frac{\varpi(\gamma-\beta)-\varpi(\gamma)}{\beta^2}d\beta d\alpha \\
& +\frac{1}{\pi}\Re\int_{\mathbb{T}}\int_{\mathbb{R}}\overline{\partial_\alpha^4 z(\gamma)}\cdot\frac{\partial_\alpha^\perp z(\gamma)}{|\partial_\alpha z(\gamma)|^4}(\partial_\alpha z(\gamma)\cdot\partial_\alpha^4 z(\gamma-\beta))\frac{\varpi(\gamma-\beta)-\varpi(\gamma)}{\beta^2}d\beta d\alpha \\
& \equiv O_7+O_8
\end{aligned}
$$

where,

$$
\begin{aligned}
O_7 = & -2\Re\int_{\mathbb{T}}\overline{\partial_\alpha^4 z(\gamma)}\cdot\frac{\partial_\alpha^\perp z(\gamma)}{|\partial_\alpha z(\gamma)|^4}(\partial_\alpha z(\gamma)\cdot\partial_\alpha^4 z(\gamma))\Lambda(\varpi)(\gamma)d\beta d\alpha \\
& \leq C\|\mathcal{F}(z)\|_{L^\infty(S)}\|\partial_\alpha^4 z\|_{L^2(S)}^2\|\Lambda\varpi\|_{L^\infty(S)}\leq C\|\mathcal{F}(z)\|_{L^\infty(S)}\|\partial_\alpha^4 z\|_{L^2(S)}^2\|\varpi\|_{\mathcal{C}^{1,\delta}(S)} \\
& \leq \exp C(\|\mathcal{F}(z)\|_{L^\infty(S)}^2+\|z\|_{H^4(S)}^2)
\end{aligned}
$$

and

$$
\begin{aligned}
O_8 & \leq \exp C(\|\mathcal{F}(z)\|_{L^\infty(S)}^2+\|z\|_{H^4(S)}^2) \\
& +\Re\int_{\mathbb{T}}\overline{\partial_\alpha^4 z(\gamma)}\cdot\frac{\partial_\alpha^\perp z(\gamma)}{|\partial_\alpha z(\gamma)|^4}(\partial_\alpha z(\gamma)\cdot H(\partial_\alpha^4 z)(\gamma))\partial_\alpha\varpi(\gamma)d\alpha \\
& \leq \exp C(\|\mathcal{F}(z)\|_{L^\infty(S)}^2+\|z\|_{H^4(S)}^2).
\end{aligned}
$$

Using integration by parts for Λ,

$$
\begin{aligned}
N_8 & = -2\Re\int_{\mathbb{T}}\Lambda(\overline{\partial_\alpha^4 z}\cdot\frac{\partial_\alpha z^\perp}{A^2(t)}\varpi\partial_\alpha z)(\gamma)\cdot\partial_\alpha^4 z(\gamma)d\alpha \\
& = -2\Re\int_{\mathbb{T}}(\Lambda(\overline{\partial_\alpha^4 z}\cdot\frac{\partial_\alpha z^\perp}{A^2(t)}\varpi\partial_\alpha z)(\gamma)-\partial_\alpha z(\gamma)\varpi(\gamma)\frac{\partial_\alpha^\perp z(\gamma)}{A^2(t)}\cdot\Lambda(\overline{\partial_\alpha^4 z})(\gamma))\cdot\partial_\alpha^4 z(\gamma)d\alpha \\
& -2\Re\int_{\mathbb{T}}\frac{\partial_\alpha^\perp z(\gamma)}{A^2(t)}\cdot\Lambda(\overline{\partial_\alpha^4 z})(\gamma)\partial_\alpha z(\gamma)\cdot\partial_\alpha^4 z(\gamma)\varpi(\gamma)d\alpha \\
& \equiv O_9+O_{10}.
\end{aligned}
$$

Using the commutator estimate (1.7),

$$
O_9\leq \exp C(\|\mathcal{F}(z)\|_{L^\infty(S)}^2+\|z\|_{H^4(S)}^2).
$$

Taking three derivatives of $A(t)=|\partial_\alpha z|^2$ we take

$$
\partial_\alpha z(\alpha)\cdot\partial_\alpha^4 z(\alpha)=-3\partial_\alpha^2 z(\alpha)\cdot\partial_\alpha^3 z(\alpha).
$$

Together with $\Lambda=\partial_\alpha H$ and integrating by parts

$$\begin{aligned}O_{10}=&-6\Re\int_{\mathbb{T}}H(\overline{\partial_\alpha^4 z})(\gamma)\cdot\frac{\partial_\alpha^2 z^\perp(\gamma)}{A^2(t)}\partial_\alpha^2 z(\gamma)\cdot\partial_\alpha^3 z(\gamma)\varpi(\gamma)d\alpha\\
&-6\Re\int_{\mathbb{T}}H(\overline{\partial_\alpha^4 z})(\gamma)\cdot\frac{\partial_\alpha^\perp z(\gamma)}{A^2(t)}\partial_\alpha^3 z(\gamma)\cdot\partial_\alpha^3 z(\gamma)\varpi(\gamma)d\alpha\\
&-6\Re\int_{\mathbb{T}}H(\overline{\partial_\alpha^4 z})(\gamma)\cdot\frac{\partial_\alpha^\perp z(\gamma)}{A^2(t)}\partial_\alpha^2 z(\gamma)\cdot\partial_\alpha^4 z(\gamma)\varpi(\gamma)d\alpha\\
&-6\Re\int_{\mathbb{T}}H(\overline{\partial_\alpha^4 z})(\gamma)\cdot\frac{\partial_\alpha^\perp z(\gamma)}{A^2(t)}\partial_\alpha^2 z(\gamma)\cdot\partial_\alpha^3 z(\gamma)\partial_\alpha\varpi(\gamma)d\alpha\\
&\leq\exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)}+\|z\|^2_{H^4(S)}).\end{aligned}$$

Then,

$$L_5\leq\exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)}+\|z\|^2_{H^4(S)}).$$

All previous discussion shows that I_3 satisfies,

$$I_3\leq\exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)}+\|z\|^2_{H^4(S)})+C\|\Im(\frac{\varpi}{A(t)})\|_{H^2(S)}\|\Lambda^{\frac{1}{2}}(\partial_\alpha^4 z)\|^2_{L^2(S)}.$$

1.1.1.2 Searching for the Raylegh-Taylor condition in I_7

Let us recall the formula for the Raylegh-Taylor condition

$$\sigma(\alpha,t)=\frac{\mu^2}{\kappa}BR(z,\varpi)(\alpha,t)\cdot\partial_\alpha^\perp z(\alpha,t)+g\rho^2\partial_\alpha z_1(\alpha,t).$$

We write I_7 in the form $I_7=K_3+K_4$ where

$$\begin{aligned}K_3&=\frac{1}{2\pi}\Re\int_{\mathbb{T}}\int_{\mathbb{R}}\overline{\partial_\alpha^4 z(\gamma)}\cdot(\frac{(z(\gamma)-z(\gamma-\beta))^\perp}{|z(\gamma)-z(\gamma-\beta)|^2}-\frac{\partial_\alpha^\perp z(\gamma)}{|\partial_\alpha z(\gamma)|^2\beta})\partial_\alpha^4\varpi(\gamma-\beta)d\beta d\alpha,\\
K_4&=\frac{1}{2\pi}\Re\int_{\mathbb{T}}\int_{\mathbb{R}}\overline{\partial_\alpha^4 z(\gamma)}\cdot\frac{\partial_\alpha^\perp z(\gamma)}{|\partial_\alpha z(\gamma)|^2\beta}\partial_\alpha^4\varpi(\gamma-\beta)d\beta d\alpha.\end{aligned}$$

After an integration by parts we obtain:

$$\begin{aligned}K_3&=-\frac{1}{2\pi}\Re\int_{\mathbb{T}}\int_{\mathbb{R}}\overline{\partial_\alpha^4 z(\gamma)}\cdot(\frac{(\Delta z)^\perp}{|\Delta z|^2}-\frac{\partial_\alpha^\perp z(\gamma)}{|\partial_\alpha z(\gamma)|^2\beta})\partial_\beta(\partial_\alpha^3\varpi(\gamma-\beta))d\beta d\alpha\\
&=\frac{1}{2\pi}\Re\int_{\mathbb{T}}\int_{\mathbb{R}}\overline{\partial_\alpha^4 z(\gamma)}\cdot\partial_\beta(\frac{(\Delta z)^\perp}{|\Delta z|^2}-\frac{\partial_\alpha^\perp z(\gamma)}{|\partial_\alpha z(\gamma)|^2\beta})\partial_\alpha^3\varpi(\gamma-\beta)d\beta d\alpha.\end{aligned}$$

We decompose

$$\begin{aligned}&\partial_\beta(\frac{(\Delta z)^\perp}{|\Delta z|^2}-\frac{\partial_\alpha^\perp z(\gamma)}{|\partial_\alpha z(\gamma)|^2\beta})\\
&=\frac{(\Delta\partial_\alpha z)^\perp}{|\Delta z|^2}+\partial_\alpha^\perp z(\gamma)(\frac{1}{|\Delta z|^2}-\frac{1}{|\partial_\alpha z(\gamma)|^2\beta^2})-2\frac{(\Delta z)^\perp\Delta z\cdot\Delta\partial_\alpha z}{|\Delta z|^4}\\
&-2\frac{(\Delta z)^\perp(\Delta z-\beta\partial_\alpha z(\gamma))\cdot\partial_\alpha z(\gamma)}{|\Delta z|^4}-2\frac{(\Delta z-\beta\partial_\alpha z(\gamma))^\perp\beta|\partial_\alpha z(\gamma)|^2}{|\Delta z|^4}\\
&+(\frac{2\partial_\alpha^\perp z(\gamma)}{|\partial_\alpha z(\gamma)|^2\beta^2}-\frac{2\beta^2\partial_\alpha^\perp z(\gamma)|\partial_\alpha z(\gamma)|^2}{|\Delta z|^4})\\
&\equiv F_1(\gamma,\beta)+F_2(\gamma,\beta)+F_3(\gamma,\beta)+F_4(\gamma,\beta)+F_5(\gamma,\beta)+F_6(\gamma,\beta)\end{aligned}\tag{1.9}$$

Then, we have

$$\begin{aligned}K_3 =& \frac{1}{2\pi}\Re\int_{\mathbb{T}}\int_{\mathbb{R}} \overline{\partial_\alpha^4 z(\gamma)}\cdot F_1(\gamma,\beta)\partial_\alpha^3\varpi(\gamma-\beta)d\beta d\alpha\\ &+\frac{1}{2\pi}\Re\int_{\mathbb{T}}\int_{\mathbb{R}} \overline{\partial_\alpha^4 z(\gamma)}\cdot F_2(\gamma,\beta)\partial_\alpha^3\varpi(\gamma-\beta)d\beta d\alpha\\ &+\frac{1}{2\pi}\Re\int_{\mathbb{T}}\int_{\mathbb{R}} \overline{\partial_\alpha^4 z(\gamma)}\cdot F_3(\gamma,\beta)\partial_\alpha^3\varpi(\gamma-\beta)d\beta d\alpha\\ &+\frac{1}{2\pi}\Re\int_{\mathbb{T}}\int_{\mathbb{R}} \overline{\partial_\alpha^4 z(\gamma)}\cdot F_4(\gamma,\beta)\partial_\alpha^3\varpi(\gamma-\beta)d\beta d\alpha\\ &+\frac{1}{2\pi}\Re\int_{\mathbb{T}}\int_{\mathbb{R}} \overline{\partial_\alpha^4 z(\gamma)}\cdot F_5(\gamma,\beta)\partial_\alpha^3\varpi(\gamma-\beta)d\beta d\alpha\\ &+\frac{1}{2\pi}\Re\int_{\mathbb{T}}\int_{\mathbb{R}} \overline{\partial_\alpha^4 z(\gamma)}\cdot F_6(\gamma,\beta)\partial_\alpha^3\varpi(\gamma-\beta)d\beta d\alpha\\ &\equiv L_6+L_7+L_8+L_9+L_{10}+L_{11}.\end{aligned}$$

For L_6, L_7, L_8, L_9 and L_{10} we can estimates with the same approach as before, and we easily get

$$L_6+L_7+L_8+L_9+L_{10}\leq \exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)}+\|z\|^2_{H^4(S)}).$$

Since,

$$\begin{aligned}&-\frac{1}{2}F_6(\gamma,\beta)=\partial_\alpha^\perp z(\gamma)\frac{\beta^4|\partial_\alpha z(\gamma)|^4-|\Delta z|^4}{|\Delta z|^4|\partial_\alpha z(\gamma)|^2\beta^2}=Q_1(\gamma,\beta)\\ &+\frac{\partial_\alpha^\perp z(\gamma)}{2}\frac{\beta^4\partial_\alpha^2 z(\gamma)\int_0^1\int_0^1\partial_\alpha^2 z(\eta)(s-1)dtds\int_0^1[|\partial_\alpha z(\gamma)|^2+|\partial_\alpha z(\phi)|^2]ds}{|\Delta z|^4|\partial_\alpha z(\gamma)|^2}\\ &+\partial_\alpha^\perp z(\gamma)\frac{\beta^4\partial_\alpha^2 z(\gamma)\partial_\alpha z(\gamma)\int_0^1\int_0^1\partial_\alpha z(\eta)\cdot\partial_\alpha^2 z(\eta)(s-1)dtds}{|\Delta z|^4|\partial_\alpha z(\gamma)|^2}+\partial_\alpha^\perp z(\gamma)\frac{\beta^3\partial_\alpha^2 z(\gamma)\partial_\alpha z(\gamma)}{|\Delta z|^4}\\ &\equiv U_1(\gamma,\beta)+U_2(\gamma,\beta)+U_3(\gamma,\beta)+U_4(\gamma,\beta)\end{aligned}$$

where $Q_1(\gamma,\beta)$ is the remainder term that does not cause any trouble. we get,

$$\begin{aligned}L_{11}=&-\frac{1}{\pi}\Re\int_{\mathbb{T}}\int_{\mathbb{R}} \overline{\partial_\alpha^4 z(\gamma)}\cdot U_1(\gamma,\beta)\partial_\alpha^3\varpi(\gamma-\beta)d\beta d\alpha\\ &-\frac{1}{\pi}\Re\int_{\mathbb{T}}\int_{\mathbb{R}} \overline{\partial_\alpha^4 z(\gamma)}\cdot U_2(\gamma,\beta)\partial_\alpha^3\varpi(\gamma-\beta)d\beta d\alpha\\ &-\frac{1}{\pi}\Re\int_{\mathbb{T}}\int_{\mathbb{R}} \overline{\partial_\alpha^4 z(\gamma)}\cdot U_3(\gamma,\beta)\partial_\alpha^3\varpi(\gamma-\beta)d\beta d\alpha\\ &-\frac{1}{\pi}\Re\int_{\mathbb{T}}\int_{\mathbb{R}} \overline{\partial_\alpha^4 z(\gamma)}\cdot U_4(\gamma,\beta)\partial_\alpha^3\varpi(\gamma-\beta)d\beta d\alpha\\ &\equiv M_9+M_{10}+M_{11}+M_{12}\end{aligned}$$

where

$$\begin{aligned}M_9&\leq C\|\mathcal{F}(z)\|^2_{L^\infty(S)}\|z\|^2_{\mathcal{C}^1(S)}\|z\|_{\mathcal{C}^3(S)}\|\partial_\alpha^4 z\|_{L^2(S)}\|\partial_\alpha^3\varpi\|_{L^2(S)},\\ M_{10}&\leq C\|\mathcal{F}(z)\|^2_{L^\infty(S)}\|z\|^2_{\mathcal{C}^2(S)}\|z\|_{\mathcal{C}^1(S)}\|\partial_\alpha^4 z\|_{L^2(S)}\|\partial_\alpha^3\varpi\|_{L^2(S)},\\ M_{11}&\leq C\|\mathcal{F}(z)\|^2_{L^\infty(S)}\|z\|_{\mathcal{C}^1(S)}\|z\|_{\mathcal{C}^2(S)}\|\partial_\alpha^4 z\|_{L^2(S)}\|\partial_\alpha^3\varpi\|_{L^2(S)}\end{aligned}$$

and if we split,

$$M_{12} = -\frac{1}{\pi}\Re\int_{\mathbb{T}}\int_{\mathbb{R}} \overline{\partial_\alpha^4 z(\gamma)}\cdot\partial_\alpha^\perp z(\gamma)\beta^3\partial_\alpha^2 z(\gamma)\partial_\alpha z(\gamma)\partial_\alpha^3\varpi(\gamma-\beta)\Big(\frac{1}{|\Delta z|^4}-\frac{1}{|\partial_\alpha z(\gamma)|^4\beta^4}\Big)d\beta d\alpha$$
$$-\frac{1}{\pi}\Re\int_{\mathbb{T}}\overline{\partial_\alpha^4 z(\gamma)}\cdot\frac{\partial_\alpha^\perp z(\gamma)\partial_\alpha^2 z(\gamma)\partial_\alpha z(\gamma)}{|\partial_\alpha z(\gamma)|^4}H(\partial_\alpha^3\varpi)(\gamma)d\alpha.$$

It is clear that

$$M_{12}\le \exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)}+\|z\|^2_{H^4(S)}).$$

Thus,

$$L_{11}\le \exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)}+\|z\|^2_{H^4(S)}).$$

Therefore,

$$K_3\le \exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)}+\|z\|^2_{H^4(S)}).$$

We consider now the K_4 term which can be written as follows

$$K_4=\frac{1}{2}\Re\int_{\mathbb{T}}\overline{\partial_\alpha^4 z(\gamma)}\cdot\frac{\partial_\alpha^\perp z(\gamma)}{|\partial_\alpha z(\gamma)|^2}H(\partial_\alpha^4\varpi)(\gamma)d\alpha=\frac{1}{2}\Re\int_{\mathbb{T}}\overline{\partial_\alpha^4 z(\gamma)}\cdot\frac{\partial_\alpha^\perp z(\gamma)}{|\partial_\alpha z(\gamma)|^2}\Lambda(\partial_\alpha^3\varpi)(\gamma)d\alpha$$
$$=\frac{1}{2}\Re\int_{\mathbb{T}}\frac{\Lambda(\overline{\partial_\alpha^4 z}\cdot\partial_\alpha z^\perp)(\gamma)}{A(t)}\partial_\alpha^3\varpi(\gamma)d\alpha$$

using the formula

$$\varpi(\alpha)=-2BR(z,\varpi)(\alpha,t)\cdot\partial_\alpha z(\alpha,t)-2\kappa g\frac{\rho^2}{\mu^2}\partial_\alpha z_2(\alpha,t)=-T(\varpi)(\alpha)-2g\kappa\frac{\rho^2}{\mu^2}\partial_\alpha z_2(\alpha)$$

we separate K_4 as a sum of two parts, L_{12} and L_{13}, where

$$L_{12}=-g\kappa\frac{\rho^2}{\mu^2}\Re\int_{\mathbb{T}}\frac{\Lambda(\overline{\partial_\alpha^4 z}\cdot\partial_\alpha z^\perp)(\gamma)}{A(t)}\partial_\alpha^4 z_2(\gamma)d\alpha,$$
$$L_{13}=-\frac{1}{2}\Re\int_{\mathbb{T}}\frac{\Lambda(\overline{\partial_\alpha^4 z}\cdot\partial_\alpha z^\perp)(\gamma)}{A(t)}\partial_\alpha^3 T(\varpi)(\gamma)d\alpha.$$

For L_{12} we decompose further $L_{12}=M_{13}+M_{14}$ where

$$M_{13}=g\kappa\frac{\rho^2}{\mu^2}\Re\int_{\mathbb{T}}\frac{\Lambda(\overline{\partial_\alpha^4 z_1}\partial_\alpha z_2)(\gamma)}{A(t)}\partial_\alpha^4 z_2(\gamma)d\alpha,$$
$$M_{14}=-g\kappa\frac{\rho^2}{\mu^2}\Re\int_{\mathbb{T}}\frac{\Lambda(\overline{\partial_\alpha^4 z_2}\partial_\alpha z_1)(\gamma)}{A(t)}\partial_\alpha^4 z_2(\gamma)d\alpha.$$

Then M_{13} is written as $M_{13}=N_9+N_{10}$ with

$$N_9=g\kappa\frac{\rho^2}{\mu^2}\Re\int_{\mathbb{T}}\frac{\Lambda(\overline{\partial_\alpha^4 z_1}\partial_\alpha z_2)(\gamma)-\partial_\alpha z_2(\gamma)\Lambda(\overline{\partial_\alpha^4 z_1})(\gamma)}{A(t)}\partial_\alpha^4 z_2(\gamma)d\alpha,$$
$$N_{10}=g\kappa\frac{\rho^2}{\mu^2}\Re\int_{\mathbb{T}}\frac{\partial_\alpha z_2(\gamma)\Lambda(\overline{\partial_\alpha^4 z_1})(\gamma)}{A(t)}\partial_\alpha^4 z_2(\gamma)d\alpha.$$

Using the commutator estimate (1.7), we get

$$N_9\le C\|\mathcal{F}(z)\|_{L^\infty(S)}\|z\|_{\mathcal{C}^{2,\delta}(S)}\|\partial_\alpha^4 z\|^2_{L^2(S)}\le \exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)}+\|z\|^2_{H^4(S)}).$$

The identity

$$\partial_\alpha z_2(\gamma)\partial_\alpha^4 z_2(\gamma) = \partial_\alpha z(\gamma)\cdot\partial_\alpha^4 z(\gamma) - \partial_\alpha z_1(\gamma)\partial_\alpha^4 z_1(\gamma)$$

let us write N_{10} as the sum of O_{11} and O_{12} where

$$\begin{aligned}
O_{11} &= g\kappa\frac{\rho^2}{\mu^2}\Re\int_{\mathbb{T}}\frac{\Lambda(\overline{\partial_\alpha^4 z_1})(\gamma)}{A(t)}\partial_\alpha z(\gamma)\cdot\partial_\alpha^4 z(\gamma)d\alpha,\\
O_{12} &= -g\kappa\frac{\rho^2}{\mu^2}\Re\int_{\mathbb{T}}\frac{\Lambda(\overline{\partial_\alpha^4 z_1})(\gamma)}{A(t)}\partial_\alpha z_1(\gamma)\partial_\alpha^4 z_1(\gamma)d\alpha.
\end{aligned}$$

For O_{11} we use an integration by parts and

$$\partial_\alpha z(\gamma)\cdot\partial_\alpha^4 z(\gamma) = -3\partial_\alpha^2 z(\gamma)\cdot\partial_\alpha^3 z(\gamma)$$

we get,

$$\begin{aligned}
O_{11} &= -3g\kappa\frac{\rho^2}{\mu^2}\Re\int_{\mathbb{T}}\frac{\Lambda(\overline{\partial_\alpha^4 z_1})(\gamma)}{A(t)}\partial_\alpha^2 z(\gamma)\cdot\partial_\alpha^3 z(\gamma)d\alpha\\
&= 3g\kappa\frac{\rho^2}{\mu^2}\Re\int_{\mathbb{T}}\frac{H(\overline{\partial_\alpha^4 z_1})(\gamma)}{A(t)}\partial_\alpha^3 z(\gamma)\cdot\partial_\alpha^3 z(\gamma)d\alpha\\
&+ 3g\kappa\frac{\rho^2}{\mu^2}\Re\int_{\mathbb{T}}\frac{H(\overline{\partial_\alpha^4 z_1})(\gamma)}{A(t)}\partial_\alpha^2 z(\gamma)\cdot\partial_\alpha^4 z(\gamma)d\alpha\\
&\leq \exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)} + \|z\|^2_{H^4(S)}).
\end{aligned}$$

And writing M_{14} in the form

$$\begin{aligned}
M_{14} = &-g\kappa\frac{\rho^2}{\mu^2}\Re\int_{\mathbb{T}}\frac{\Lambda(\overline{\partial_\alpha^4 z_2}\cdot\partial_\alpha z_1)(\gamma) - \partial_\alpha z_1(\gamma)\Lambda(\overline{\partial_\alpha^4 z_2})(\gamma)}{A(t)}\partial_\alpha^4 z_2(\gamma)d\alpha\\
&- g\kappa\frac{\rho^2}{\mu^2}\Re\int_{\mathbb{T}}\frac{\partial_\alpha z_1(\gamma)}{A(t)}\Lambda(\overline{\partial_\alpha^4 z_2})(\gamma)\partial_\alpha^4 z_2(\gamma)d\alpha \equiv N_{11} + N_{12},
\end{aligned}$$

by the commutator estimate, we have

$$N_{11} \leq C\|\mathcal{F}(z)\|_{L^\infty(S)}\|z\|_{\mathcal{C}^{2,\delta}(S)}\|\partial_\alpha^4 z\|^2_{L^2(S)} \leq \exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)} + \|z\|^2_{H^4(S)}).$$

Since,

$$O_{12} + N_{12} = -g\kappa\frac{\rho^2}{\mu^2}\Re\int_{\mathbb{T}}\frac{\partial_\alpha z_1(\gamma)}{A(t)}\partial_\alpha^4 z(\gamma)\cdot\Lambda(\overline{\partial_\alpha^4 z})(\gamma)d\alpha$$

we obtain finally

$$L_{12} \leq \exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)} + \|z\|^2_{H^4(S)}) - g\kappa\frac{\rho^2}{\mu^2}\Re\int_{\mathbb{T}}\frac{\partial_\alpha z_1(\gamma)}{A(t)}\partial_\alpha^4 z(\gamma)\cdot\Lambda(\overline{\partial_\alpha^4 z})(\gamma)d\alpha.$$

In the estimate above we can observe how part of $\sigma(\gamma)$ appears in the non-integrable terms. Let us return to $L_{13} = M_{15} + M_{16} + M_{17} + M_{18}$ where

$$M_{15} = -\Re\int_{\mathbb{T}}\frac{\Lambda(\overline{\partial_\alpha^4 z}\cdot\partial_\alpha z^\perp)(\gamma)}{A(t)}\partial_\alpha^3 BR(z,\varpi)(\gamma)\cdot\partial_\alpha z(\gamma)d\alpha,$$

$$M_{16} = -3\Re\int_{\mathbb{T}} \frac{\Lambda(\overline{\partial_\alpha^4 z}\cdot\partial_\alpha z^\perp)(\gamma)}{A(t)}\partial_\alpha^2 BR(z,\varpi)(\gamma)\cdot\partial_\alpha^2 z(\gamma)d\alpha,$$
$$M_{17} = -3\Re\int_{\mathbb{T}} \frac{\Lambda(\overline{\partial_\alpha^4 z}\cdot\partial_\alpha z^\perp)(\gamma)}{A(t)}\partial_\alpha BR(z,\varpi)(\gamma)\cdot\partial_\alpha^3 z(\gamma)d\alpha,$$
$$M_{18} = -\Re\int_{\mathbb{T}} \frac{\Lambda(\overline{\partial_\alpha^4 z}\cdot\partial_\alpha z^\perp)(\gamma)}{A(t)}BR(z,\varpi)(\gamma)\cdot\partial_\alpha^4 z(\gamma)d\alpha.$$

We will control first the terms M_{16}, M_{15} and M_{17} and then we will show how the rest of $\sigma(\gamma)$ appears in M_{18}. Using $\Lambda = H\partial_\alpha$ and integrating by parts, we obtain

$$\begin{aligned} M_{16} &= 3\Re\int_{\mathbb{T}} \frac{H(\overline{\partial_\alpha^4 z}\cdot\partial_\alpha z^\perp)(\gamma)}{A(t)}\partial_\alpha^3 BR(z,\varpi)(\gamma)\cdot\partial_\alpha^2 z(\gamma)d\alpha \\ &\quad + 3\Re\int_{\mathbb{T}} \frac{H(\overline{\partial_\alpha^4 z}\cdot\partial_\alpha z^\perp)(\gamma)}{A(t)}\partial_\alpha^2 BR(z,\varpi)(\gamma)\cdot\partial_\alpha^3 z(\gamma)d\alpha \\ &\equiv N_{13} + N_{14}. \end{aligned}$$

With (1.4)

$$\begin{aligned} N_{13} &\leq C\|\mathcal{F}(z)\|_{L^\infty(S)}\|z\|_{\mathcal{C}^2(S)}\|\partial_\alpha^3 BR\|_{L^2(S)}\|\partial_\alpha^4 z\cdot\partial_\alpha z\|_{L^2(S)} \\ &\leq \exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)} + \|z\|^2_{H^4(S)}) \end{aligned}$$

and

$$\begin{aligned} N_{14} &\leq C\|\mathcal{F}(z)\|_{L^\infty(S)}\|z\|_{\mathcal{C}^3(S)}\|\partial_\alpha^2 BR\|_{L^2(S)}\|\partial_\alpha^4 z\cdot\partial_\alpha z\|_{L^2(S)} \\ &\leq \exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)} + \|z\|^2_{H^4(S)}). \end{aligned}$$

With M_{15} we also integrate by parts to obtain $M_{15} = N_{15} + N_{16}$ where

$$N_{15} = \Re\int_{\mathbb{T}} \frac{H(\overline{\partial_\alpha^4 z}\cdot\partial_\alpha z^\perp)(\gamma)}{A(t)}\partial_\alpha^4 BR(z,\varpi)(\gamma)\cdot\partial_\alpha z(\gamma)d\alpha,$$
$$N_{16} = \Re\int_{\mathbb{T}} \frac{H(\overline{\partial_\alpha^4 z}\cdot\partial_\alpha z^\perp)(\gamma)}{A(t)}\partial_\alpha^3 BR(z,\varpi)(\gamma)\cdot\partial_\alpha^2 z(\gamma)d\alpha.$$

Easily we have

$$\begin{aligned} N_{16} &\leq C\|\mathcal{F}(z)\|_{L^\infty(S)}\|z\|_{\mathcal{C}^2(S)}\|\partial_\alpha^4 z\cdot\partial_\alpha z\|_{L^2(S)}\|\partial_\alpha^3 BR\|_{L^2(S)} \\ &\leq \exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)} + \|z\|^2_{H^4(S)}). \end{aligned}$$

In N_{15} the application of Leibniz's rule to $\partial_\alpha^3 BR(z,\varpi)$ produces many terms which can be estimated with the same tools used before. For the most singular terms we have the expressions:

$$O_{13} = \Re\int_{\mathbb{T}} \frac{H(\overline{\partial_\alpha^4 z}\cdot\partial_\alpha z^\perp)(\gamma)}{A(t)}\partial_\alpha BR(z,\partial_\alpha^3\varpi)(\gamma)\cdot\partial_\alpha z(\gamma)d\alpha,$$
$$O_{14} = \Re\int_{\mathbb{T}} \frac{H(\overline{\partial_\alpha^4 z}\cdot\partial_\alpha z^\perp)(\gamma)}{A(t)}\int_{\mathbb{T}} \frac{\Delta\partial_\alpha^4 z}{|\Delta z|^2}\cdot\partial_\alpha z(\gamma)\varpi(\gamma-\beta)d\beta d\alpha,$$

$$O_{15} = -2\Re\int_{\mathbb{T}} \frac{H(\overline{\partial_\alpha^4 z}\cdot\partial_\alpha z^\perp)(\gamma)}{A(t)}\int_{\mathbb{T}}\frac{\Delta z^\perp\cdot\partial_\alpha z(\gamma)}{|\Delta z|^4}(\Delta z\cdot\Delta\partial_\alpha^4 z)\varpi(\gamma-\beta)d\beta d\alpha.$$

Let us consider

$$\begin{aligned}\partial_\alpha BR(z,\varpi)(\gamma)\cdot\partial_\alpha z(\gamma) &= \partial_\alpha(BR(z,\varpi)(\gamma)\cdot\partial_\alpha z(\gamma)) - BR(z,\varpi)(\gamma)\cdot\partial_\alpha^2 z(\gamma)\\ &= \frac{1}{2}\partial_\alpha T(\varpi)(\gamma) - BR(z,\varpi)\cdot\partial_\alpha^2 z(\gamma)\end{aligned}$$

which yields

$$\begin{aligned}O_{13} &\le C\|\mathcal{F}(z)\|_{L^\infty(S)}\|\partial_\alpha^4 z\|_{L^2(S)}\|z\|_{\mathcal{C}^1(S)}(\|T(\partial_\alpha^3\varpi)\|_{H^1(S)} + \|BR(z,\partial_\alpha^3\varpi)\|_{L^2(S)}\|z\|_{\mathcal{C}^2(S)})\\ &\le \exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)} + \|z\|^2_{H^4(S)})\end{aligned}$$

because $\|T\|_{L^2\to H^1} \le \|\mathcal{F}(z)\|^2_{L^\infty}\|z\|^4_{\mathcal{C}^{2,\delta}}$ for more details see Lemma 3.1 in [11]. Next we write $O_{14} = P_3 + P_4$,

$$\begin{aligned}P_3 &= \Re\int_{\mathbb{T}}\int_{\mathbb{R}}\frac{H(\overline{\partial_\alpha^4 z}\cdot\partial_\alpha z^\perp)(\gamma)}{A(t)}\Delta\partial_\alpha^4 z\cdot\partial_\alpha z(\gamma)\varpi(\gamma-\beta)(\frac{1}{|\Delta z|^2} - \frac{1}{|\partial_\alpha z(\gamma)|^2\beta^2})d\beta d\alpha,\\ P_4 &= \Re\int_{\mathbb{T}}\int_{\mathbb{R}}\frac{H(\overline{\partial_\alpha^4 z}\cdot\partial_\alpha z^\perp)(\gamma)}{A(t)}\Delta\partial_\alpha^4 z\cdot\partial_\alpha z(\gamma)\frac{\varpi(\gamma-\beta)}{A(t)\beta^2}d\beta d\alpha.\end{aligned}$$

Using $B_2(\gamma,\beta) = B_3(\gamma,\beta) + B_4(\gamma,\beta)$, we split $P_3 = Q_1 + Q_2 + Q_3$,

$$\begin{aligned}Q_1 &= \Re\int_{\mathbb{T}}\int_{\mathbb{R}}\frac{H(\overline{\partial_\alpha^4 z}\cdot\partial_\alpha z^\perp)(\gamma)}{A(t)}\Delta\partial_\alpha^4 z\cdot\partial_\alpha z(\gamma)\varpi(\gamma-\beta)B_1(\gamma,\beta)d\beta d\alpha,\\ Q_2 &= \Re\int_{\mathbb{T}}\int_{\mathbb{R}}\frac{H(\overline{\partial_\alpha^4 z}\cdot\partial_\alpha z^\perp)(\gamma)}{A(t)}\Delta\partial_\alpha^4 z\cdot\partial_\alpha z(\gamma)\varpi(\gamma-\beta)B_3(\gamma,\beta)d\beta d\alpha,\\ Q_3 &= \Re\int_{\mathbb{T}}\int_{\mathbb{R}}\frac{H(\overline{\partial_\alpha^4 z}\cdot\partial_\alpha z^\perp)(\gamma)}{A(t)}\Delta\partial_\alpha^4 z\cdot\partial_\alpha z(\gamma)\varpi(\gamma-\beta)B_4(\gamma,\beta)d\beta d\alpha.\end{aligned}$$

being

$$\begin{aligned}B_1(\gamma,\beta) &= \frac{\beta\int_0^1\int_0^1\frac{\partial_\alpha^2 z(\psi)-\partial_\alpha^2 z(\gamma)}{|\psi-\gamma|^\delta}\beta^\delta(1+s+t-st)^\delta(1-s)dtds\int_0^1[\partial_\alpha z(\gamma)+\partial_\alpha z(\phi)]ds}{|z(\gamma)-z(\gamma-\beta)|^2|\partial_\alpha z(\gamma)|^2},\\ B_3(\gamma,\beta) &= \frac{\beta^2\partial_\alpha^2 z(\gamma)\int_0^1\int_0^1\partial_\alpha^2 z(\eta)(s-1)dtds}{|z(\gamma)-z(\gamma-\beta)|^2|\partial_\alpha z(\gamma)|^2},\\ B_4(\gamma,\beta) &= \frac{\beta\partial_\alpha^2 z(\gamma)2\partial_\alpha z(\gamma)}{|z(\gamma)-z(\gamma-\beta)|^2|\partial_\alpha z(\gamma)|^2}\end{aligned}$$

therefore

$$\begin{aligned}Q_1 &\le C\|\mathcal{F}(z)\|^2_{L^\infty(S)}\|z\|_{\mathcal{C}^1(S)}\|z\|_{\mathcal{C}^{2,\delta}(S)}\|\varpi\|_{L^\infty(S)}\|\partial_\alpha^4 z\|^2_{L^2(S)},\\ Q_2 &\le C\|\mathcal{F}(z)\|^2_{L^\infty(S)}\|z\|_{\mathcal{C}^1(S)}\|z\|^2_{\mathcal{C}^2(S)}\|\varpi\|_{L^\infty(S)}\|\partial_\alpha^4 z\|^2_{L^2(S)}\end{aligned}$$

and

$$Q_3 = 2\Re\int_{\mathbb{T}}\int_{\mathbb{R}}\frac{H(\overline{\partial_\alpha^4 z}\cdot\partial_\alpha z^\perp)(\gamma)}{A^2(t)}\Delta\partial_\alpha^4 z\cdot\partial_\alpha z(\gamma)\varpi(\gamma-\beta)\beta\partial_\alpha^2 z(\gamma)\partial_\alpha z(\gamma)B(\gamma,\beta)d\beta d\alpha$$

$$+\,2\Re\int_{\mathbb{T}}\int_{\mathbb{R}}\frac{H(\overline{\partial_\alpha^4 z}\cdot\partial_\alpha z^\perp)(\gamma)}{A^2(t)}\Delta\partial_\alpha^4 z\cdot\partial_\alpha z(\gamma)\varpi(\gamma-\beta)\frac{\partial_\alpha^2 z(\gamma)\partial_\alpha z(\gamma)}{|\partial_\alpha z(\gamma)|^2\beta}d\beta d\alpha\equiv R_1+R_2.$$

Recall that

$$B(\gamma,\beta)\equiv\frac{\beta\int_0^1\int_0^1\partial_\alpha^2 z(\psi)(1-s)dtds\int_0^1\partial_\alpha z(\gamma)+\partial_\alpha z(\phi)ds}{|z(\gamma)-z(\gamma-\beta)|^2|\partial_\alpha z(\gamma)|^2},$$

then the term R_1 is controlled. We split $R_2=S_1+S_2$ where

$$\begin{aligned}S_1&=2\Re\int_{\mathbb{T}}\int_{\mathbb{R}}\frac{H(\overline{\partial_\alpha^4 z}\cdot\partial_\alpha z^\perp)(\gamma)}{A^2(t)}\partial_\alpha^4 z(\gamma)\cdot\partial_\alpha z(\gamma)\varpi(\gamma-\beta)\frac{\partial_\alpha^2 z(\gamma)\partial_\alpha z(\gamma)}{|\partial_\alpha z(\gamma)|^2\beta}d\beta d\alpha,\\ S_2&=-2\Re\int_{\mathbb{T}}\int_{\mathbb{R}}\frac{H(\overline{\partial_\alpha^4 z}\cdot\partial_\alpha z^\perp)(\gamma)}{A^2(t)}\partial_\alpha^4 z(\gamma-\beta)\cdot\partial_\alpha z(\gamma)\varpi(\gamma-\beta)\frac{\partial_\alpha^2 z(\gamma)\partial_\alpha z(\gamma)}{|\partial_\alpha z(\gamma)|^2\beta}d\beta d\alpha.\end{aligned}$$

Easily

$$\begin{aligned}S_1&=2\Re\int_{\mathbb{T}}\frac{H(\overline{\partial_\alpha^4 z}\cdot\partial_\alpha z^\perp)(\gamma)}{A^2(t)}\partial_\alpha^4 z(\gamma)\cdot\partial_\alpha z(\gamma)H(\varpi)(\gamma)\frac{\partial_\alpha^2 z(\gamma)\partial_\alpha z(\gamma)}{|\partial_\alpha z(\gamma)|^2}d\alpha\\ &\le C\|\mathcal{F}(z)\|^2_{L^\infty(S)}\|\partial_\alpha^4 z\|^2_{L^2(S)}\|z\|_{\mathcal{C}^1(S)}\|z\|_{\mathcal{C}^2(S)}\|H\varpi\|_{L^\infty(S)}\\ &\le\exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)}+\|z\|^2_{H^4(S)})\end{aligned}$$

and

$$\begin{aligned}S_2&\le C\|\mathcal{F}(z)\|^2_{L^\infty(S)}\|\partial_\alpha^4 z\|^2_{L^2(S)}\|z\|_{\mathcal{C}^1(S)}\|z\|_{\mathcal{C}^2(S)}\|\varpi\|_{\mathcal{C}^1(S)}\\ &-2\Re\int_{\mathbb{T}}\frac{H(\overline{\partial_\alpha^4 z}\cdot\partial_\alpha z^\perp)(\gamma)}{A^2(t)}H(\partial_\alpha^4 z)(\gamma)\cdot\partial_\alpha z(\gamma)\varpi(\gamma)\frac{\partial_\alpha^2 z(\gamma)\partial_\alpha z(\gamma)}{|\partial_\alpha z(\gamma)|^2}d\alpha\\ &\le\exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)}+\|z\|^2_{H^4(S)})\end{aligned}$$

Hence,

$$P_3\le\exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)}+\|z\|^2_{H^4(S)}).$$

We decompose $P_4=Q_4+Q_5$,

$$\begin{aligned}Q_4&=\Re\int_{\mathbb{T}}\int_{\mathbb{R}}\frac{H(\overline{\partial_\alpha^4 z}\cdot\partial_\alpha z^\perp)(\gamma)}{A^2(t)}\Delta\partial_\alpha^4 z\cdot\partial_\alpha z(\gamma)\frac{\varpi(\gamma-\beta)-\varpi(\gamma)}{\beta^2}d\beta d\alpha,\\ Q_5&=\Re\int_{\mathbb{T}}\int_{\mathbb{R}}\frac{H(\overline{\partial_\alpha^4 z}\cdot\partial_\alpha z^\perp)(\gamma)}{A^2(t)}\Delta\partial_\alpha^4 z\cdot\partial_\alpha z(\gamma)\frac{\varpi(\gamma)}{\beta^2}d\beta d\alpha.\end{aligned}$$

Then we split

$$\begin{aligned}Q_4&=\Re\int_{\mathbb{T}}\int_{\mathbb{R}}\frac{H(\overline{\partial_\alpha^4 z}\cdot\partial_\alpha z^\perp)(\gamma)}{A^2(t)}\partial_\alpha^4 z(\gamma)\cdot\partial_\alpha z(\gamma)\frac{\varpi(\gamma-\beta)-\varpi(\gamma)}{\beta^2}d\beta d\alpha\\ &-\Re\int_{\mathbb{T}}\int_{\mathbb{R}}\frac{H(\overline{\partial_\alpha^4 z}\cdot\partial_\alpha z^\perp)(\gamma)}{A^2(t)}\partial_\alpha^4 z(\gamma-\beta)\cdot\partial_\alpha z(\gamma)\frac{\varpi(\gamma-\beta)-\varpi(\gamma)}{\beta^2}d\beta d\alpha\\ &\equiv R_3+R_4\end{aligned}$$

where

$$R_3\le\exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)}+\|z\|^2_{H^4(S)})$$

and,

$$
\begin{aligned}
R_4 &\leq C\|\mathcal{F}(z)\|^{\frac{3}{2}}_{L^\infty(S)}\|\partial^4_\alpha z\|^2_{L^2(S)}\|z\|_{\mathcal{C}^1(S)}\|\varpi\|_{\mathcal{C}^2(S)}\\
&\quad -\pi\Re\int_{\mathbb{T}}\frac{H(\overline{\partial^4_\alpha z}\cdot\partial_\alpha z^\perp)(\gamma)}{A^2(t)}H(\partial^4_\alpha z)(\gamma)\cdot\partial_\alpha z(\gamma)\partial_\alpha\varpi(\gamma)d\alpha\\
&\leq \exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)}+\|z\|^2_{H^4(S)}).
\end{aligned}
$$

Then,

$$
Q_4 \leq \exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)}+\|z\|^2_{H^4(S)}).
$$

For Q_5 integrating by parts for Λ we have,

$$
\begin{aligned}
Q_5 &= \Re\int_{\mathbb{T}}\Lambda(\frac{H(\overline{\partial^4_\alpha z}\cdot\partial_\alpha z^\perp)}{A^2(t)}\partial_\alpha z\varpi)\cdot\partial^4_\alpha z(\gamma)d\alpha\\
&= \Re\int_{\mathbb{T}}(\Lambda(\frac{H(\overline{\partial^4_\alpha z}\cdot\partial_\alpha z^\perp)}{A^2(t)}\partial_\alpha z\varpi)(\gamma)-\partial_\alpha z(\gamma)\Lambda(\frac{H(\overline{\partial^4_\alpha z}\cdot\partial_\alpha z^\perp)}{A^2(t)})(\gamma))\cdot\partial^4_\alpha z(\gamma)d\alpha\\
&+\Re\int_{\mathbb{T}}\partial_\alpha z(\gamma)\Lambda(\frac{H(\overline{\partial^4_\alpha z}\cdot\partial_\alpha z^\perp)}{A^2(t)})(\gamma)\cdot\partial^4_\alpha z(\gamma)d\alpha\\
&\leq \exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)}+\|z\|^2_{H^4(S)})-\Re\int_{\mathbb{T}}\partial_\alpha z(\gamma)\frac{\partial_\alpha(\overline{\partial^4_\alpha z}\cdot\partial_\alpha z^\perp)(\gamma)}{A^2(t)}\cdot\partial^4_\alpha z(\gamma)d\alpha
\end{aligned}
$$

now, using $\partial_\alpha z(\alpha)\cdot\partial^4_\alpha z(\alpha)=-3\partial^2_\alpha z(\alpha)\cdot\partial^3_\alpha z(\alpha)$

$$
\begin{aligned}
&-\Re\int_{\mathbb{T}}\frac{\partial_\alpha(\overline{\partial^4_\alpha z}\cdot\partial_\alpha z^\perp)(\gamma)}{A^2(t)}\partial_\alpha z(\gamma)\cdot\partial^4_\alpha z(\gamma)d\alpha = 3\Re\int_{\mathbb{T}}\frac{\partial_\alpha(\overline{\partial^4_\alpha z}\cdot\partial_\alpha z^\perp)(\gamma)}{A^2(t)}\partial^2_\alpha z(\gamma)\cdot\partial^3_\alpha z(\gamma)d\alpha\\
&= -3\Re\int_{\mathbb{T}}\frac{\overline{\partial^4_\alpha z}\cdot\partial_\alpha z^\perp(\gamma)}{A^2(t)}\partial^3_\alpha z(\gamma)\cdot\partial^3_\alpha z(\gamma)d\alpha-3\Re\int_{\mathbb{T}}\frac{\overline{\partial^4_\alpha z}\cdot\partial_\alpha z^\perp(\gamma)}{A^2(t)}\partial^2_\alpha z(\gamma)\cdot\partial^4_\alpha z(\gamma)d\alpha\\
&\leq \exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)}+\|z\|^2_{H^4(S)}).
\end{aligned}
$$

Therefore,

$$
P_4 \leq \exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)}+\|z\|^2_{H^4(S)}).
$$

Thus, O_{14} satisfies identical estimates than P_4. To conclude with N_{15}, let us estimate O_{15}. We split $O_{15}=P_5+P_6$

$$
\begin{aligned}
P_5 &= -2\Re\int_{\mathbb{T}}\int_{\mathbb{R}}\frac{H(\overline{\partial^4_\alpha z}\cdot\partial_\alpha z^\perp)(\gamma)}{A(t)}C(\gamma,\beta)\cdot\partial_\alpha z(\gamma)(\Delta z\cdot\Delta\partial^4_\alpha z)\varpi(\gamma-\beta)d\beta d\alpha,\\
P_6 &= -2\Re\int_{\mathbb{T}}\int_{\mathbb{R}}\frac{H(\overline{\partial^4_\alpha z}\cdot\partial_\alpha z^\perp)(\gamma)}{A(t)}\frac{\partial^\perp_\alpha z(\gamma)}{A^2(t)\beta^3}\cdot\partial_\alpha z(\gamma)(\Delta z\cdot\Delta\partial^4_\alpha z)\varpi(\gamma-\beta)d\beta d\alpha.
\end{aligned}
$$

Since $\partial^\perp_\alpha z(\gamma)\cdot\partial_\alpha z(\gamma)=0$, for (1.8) we have $C(\gamma,\beta)\cdot\partial_\alpha z(\gamma)=C_1(\gamma,\beta)\cdot\partial_\alpha z(\gamma)$ and $P_6=0$. Recall that,

$$
C_1(\gamma,\beta)=\frac{\beta^2\int_0^1\int_0^1\partial^2_\alpha z^\perp(\eta)(s-1)dsdt}{|\Delta z|^4}
$$

with $\eta=\gamma-t\beta+st\beta$. Using

$$\Delta\partial_\alpha^k z=\beta\int_0^1\partial_\alpha^{k+1}z(\phi)ds$$

and

$$C_1(\gamma,\beta)\cdot\partial_\alpha z(\gamma)-\frac{\beta^2\partial_\alpha^2 z^\perp(\gamma)\cdot\partial_\alpha z(\gamma)}{|\Delta z|^4}=\frac{\beta^2\int_0^1\int_0^1[\partial_\alpha^2 z^\perp(\eta)-\partial_\alpha^2 z(\gamma)](s-1)dtds\cdot\partial_\alpha z(\gamma)}{|\Delta z|^4}$$

we get

$$\begin{aligned}
O_{15}\le&\exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)}+\|z\|^2_{H^4(S)})\\
&-2\Re\int_{\mathbb{T}}\int_{\mathbb{R}}\frac{H(\overline{\partial_\alpha^4 z}\cdot\partial_\alpha z^\perp)(\gamma)}{A^3(t)}\partial_\alpha^2 z^\perp(\gamma)\cdot\partial_\alpha z(\gamma)\partial_\alpha z(\gamma)\cdot\Delta\partial_\alpha^4 z\frac{\varpi(\gamma-\beta)}{\beta}d\beta d\alpha\\
\le&\exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)}+\|z\|^2_{H^4(S)})\\
&-2\pi\Re\int_{\mathbb{T}}\frac{H(\overline{\partial_\alpha^4 z}\cdot\partial_\alpha z^\perp)(\gamma)}{A^3(t)}\partial_\alpha^2 z^\perp(\gamma)\cdot\partial_\alpha z(\gamma)\partial_\alpha z(\gamma)\cdot\partial_\alpha^4 z(\gamma)H(\varpi)(\gamma)d\alpha\\
&-4\pi\Re\int_{\mathbb{T}}\frac{H(\overline{\partial_\alpha^4 z}\cdot\partial_\alpha z^\perp)(\gamma)}{A^3(t)}\partial_\alpha^2 z^\perp(\gamma)\cdot\partial_\alpha z(\gamma)\partial_\alpha z(\gamma)\cdot H(\partial_\alpha^4 z)(\gamma)\varpi(\gamma)d\alpha.
\end{aligned}$$

Therefore we can control O_{15}. Let us decompose

$$\begin{aligned}
M_{17}=&3\Re\int_{\mathbb{T}}\frac{H(\overline{\partial_\alpha^4 z}\cdot\partial_\alpha z^\perp)(\gamma)}{A(t)}\partial_\alpha^2 BR(z,\varpi)(\gamma)\cdot\partial_\alpha^3 z(\gamma)d\alpha\\
&+3\Re\int_{\mathbb{T}}\frac{H(\overline{\partial_\alpha^4 z}\cdot\partial_\alpha z^\perp)(\gamma)}{A(t)}\partial_\alpha BR(z,\varpi)(\gamma)\cdot\partial_\alpha^4 z(\gamma)d\alpha\equiv N_{17}+N_{18},
\end{aligned}$$

using (1.4)

$$\begin{aligned}
N_{17}&\le C\|\mathcal{F}(z)\|_{L^\infty(S)}\|z\|_{\mathcal{C}^3(S)}\|BR\|_{H^2(S)}\|z\|_{\mathcal{C}^1(S)}\|\partial_\alpha^4 z\|_{L^2(S)}\le\exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)}+\|z\|^2_{H^4(S)}),\\
N_{18}&\le C\|\mathcal{F}(z)\|_{L^\infty(S)}\|\partial_\alpha BR\|_{L^\infty(S)}\|z\|_{\mathcal{C}^1(S)}\|\partial_\alpha^4 z\|^2_{L^2(S)}\le\exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)}+\|z\|^2_{H^4(S)}).
\end{aligned}$$

Then $M_{17}\le\exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)}+\|z\|^2_{H^4(S)})$. Finally we have to find the rest of $\sigma(\gamma)$ in M_{18}. To do that let us split $M_{18}=N_{19}+N_{20}+N_{21}+N_{22}$ where

$$\begin{aligned}
N_{19}&=\Re\int_{\mathbb{T}}\frac{\Lambda(\overline{\partial_\alpha^4 z_1}\partial_\alpha z_2)(\gamma)}{A(t)}BR_1(z,\varpi)(\gamma)\partial_\alpha^4 z_1(\gamma)d\alpha,\\
N_{20}&=\Re\int_{\mathbb{T}}\frac{\Lambda(\overline{\partial_\alpha^4 z_1}\partial_\alpha z_2)(\gamma)}{A(t)}BR_2(z,\varpi)(\gamma)\partial_\alpha^4 z_2(\gamma)d\alpha,\\
N_{21}&=-\Re\int_{\mathbb{T}}\frac{\Lambda(\overline{\partial_\alpha^4 z_2}\partial_\alpha z_1)(\gamma)}{A(t)}BR_1(z,\varpi)(\gamma)\partial_\alpha^4 z_1(\gamma)d\alpha,\\
N_{22}&=-\Re\int_{\mathbb{T}}\frac{\Lambda(\overline{\partial_\alpha^4 z_2}\partial_\alpha z_1)(\gamma)}{A(t)}BR_2(z,\varpi)(\gamma)\partial_\alpha^4 z_2(\gamma)d\alpha.
\end{aligned}$$

Then

$$\begin{aligned}
N_{19}=&\Re\int_{\mathbb{T}}\frac{\Lambda(\overline{\partial_\alpha^4 z_1}\partial_\alpha z_2)(\gamma)-\partial_\alpha z_2(\gamma)\Lambda(\overline{\partial_\alpha^4 z_1})(\gamma)}{A(t)}BR_1(z,\varpi)(\gamma)\partial_\alpha^4 z_1(\gamma)d\alpha\\
&+\Re\int_{\mathbb{T}}\frac{\partial_\alpha z_2(\gamma)\Lambda(\overline{\partial_\alpha^4 z_1})(\gamma)}{A(t)}BR_1(z,\varpi)(\gamma)\partial_\alpha^4 z_1(\gamma)d\alpha,
\end{aligned}$$

and the commutator estimates yields

$$N_{19} \leq \exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)} + \|z\|^2_{H^4(S)}) + \Re\int_{\mathbb{T}} \frac{\partial_\alpha z_2(\gamma) BN_9(z,\varpi)(\gamma)}{A(t)} \partial^4_\alpha z_1(\gamma)\Lambda(\overline{\partial^4_\alpha z_1})(\gamma)d\alpha.$$

In a similar way we have

$$N_{20} \leq \exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)} + \|z\|^2_{H^4(S)}) + \Re\int_{\mathbb{T}} \frac{\partial_\alpha z_2(\gamma) BR_2(z,\varpi)(\gamma)}{A(t)} \partial^4_\alpha z_2(\gamma)\Lambda(\overline{\partial^4_\alpha z_1})(\gamma)d\alpha,$$
$$N_{21} \leq \exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)} + \|z\|^2_{H^4(S)}) - \Re\int_{\mathbb{T}} \frac{\partial_\alpha z_1(\gamma) BR_1(z,\varpi)(\gamma)}{A(t)} \partial^4_\alpha z_1(\gamma)\Lambda(\overline{\partial^4_\alpha z_2})(\gamma)d\alpha,$$
$$N_{22} \leq \exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)} + \|z\|^2_{H^4(S)}) - \Re\int_{\mathbb{T}} \frac{\partial_\alpha z_1(\gamma) BR_2(z,\varpi)(\gamma)}{A(t)} \partial^4_\alpha z_2(\gamma)\Lambda(\overline{\partial^4_\alpha z_2})(\gamma)d\alpha.$$

Since,

$$\partial_\alpha z_2 \partial^4_\alpha z_2 = \partial_\alpha z \cdot \partial^4_\alpha z - \partial_\alpha z_1 \partial^4_\alpha z_1 = -3\partial^2_\alpha z \cdot \partial^3_\alpha z - \partial_\alpha z_1 \partial^4_\alpha z_1$$

and $H\partial_\alpha = \Lambda$, using integration by parts

$$\begin{aligned}
&N_{20} \leq \exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)} + \|z\|^2_{H^4(S)}) - 3\Re\int_{\mathbb{T}} \partial_\alpha(\frac{BR_2(z,\varpi)(\gamma)}{A(t)}\partial^2_\alpha z\cdot\partial^3_\alpha z) H(\overline{\partial^4_\alpha z_1})(\gamma)d\alpha \\
&- \Re\int_{\mathbb{T}} \frac{\partial_\alpha z_1(\gamma) BR_2(z,\varpi)(\gamma)}{A(t)} \partial^4_\alpha z_1(\gamma)\Lambda(\overline{\partial^4_\alpha z_1})(\gamma)d\alpha \\
&\leq \exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)} + \|z\|^2_{H^4(S)}) - \Re\int_{\mathbb{T}} \frac{\partial_\alpha z_1(\gamma) BR_2(z,\varpi)(\gamma)}{A(t)} \partial^4_\alpha z_1(\gamma)\Lambda(\overline{\partial^4_\alpha z_1})(\gamma)d\alpha.
\end{aligned}$$

In the same way,

$$N_{21} \leq \exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)} + \|z\|^2_{H^4(S)}) + \Re\int_{\mathbb{T}} \frac{\partial_\alpha z_2(\gamma) BR_1(z,\varpi)(\gamma)}{A(t)} \partial^4_\alpha z_2(\gamma)\Lambda(\overline{\partial^4_\alpha z_2})(\gamma)d\alpha.$$

Therefore,

$$\begin{aligned}
&N_{19} + N_{20} + N_{21} + N_{22} \leq \exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)} + \|z\|^2_{H^4(S)}) \\
&- \Re\int_{\mathbb{T}} \frac{BR(z,\varpi)(\gamma)\cdot\partial^\perp_\alpha z(\gamma)}{A(t)} \partial^4_\alpha z(\gamma)\cdot\Lambda(\overline{\partial^4_\alpha z})(\gamma)d\alpha.
\end{aligned}$$

Then,

$$\begin{aligned}
&L_{12} + L_{13} \leq \exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)} + \|z\|^2_{H^4(S)}) \\
&- \Re\int_{\mathbb{T}} \frac{BR(z,\varpi)(\gamma)\cdot\partial_\alpha z^\perp(\gamma)}{A(t)} \partial^4_\alpha z(\gamma)\cdot\Lambda(\overline{\partial^4_\alpha z})(\gamma)d\alpha - g\kappa\frac{\rho^2}{\mu^2}\Re\int_{\mathbb{T}} \frac{\partial_\alpha z_1(\gamma)}{A(t)} \partial^4_\alpha z(\gamma)\cdot\Lambda(\overline{\partial^4_\alpha z})(\gamma)d\alpha.
\end{aligned}$$

Let us look at these last two terms,

$$\begin{aligned}
&- \Re\int_{\mathbb{T}} \frac{BR(z,\varpi)(\gamma)\cdot\partial_\alpha z^\perp(\gamma)}{A(t)} \partial^4_\alpha z(\gamma)\cdot\Lambda(\overline{\partial^4_\alpha z})(\gamma)d\alpha - \kappa\frac{\rho^2}{\mu^2}\Re\int_{\mathbb{T}} \frac{\partial_\alpha z_1(\gamma)}{A(t)} \partial^4_\alpha z(\gamma)\cdot\Lambda(\overline{\partial^4_\alpha z})(\gamma)d\alpha \\
&= -\Re\int_{\mathbb{T}} \frac{\sigma(\gamma,t)}{A(t)} \partial^4_\alpha z(\gamma)\cdot\Lambda(\overline{\partial^4_\alpha z})(\gamma)d\alpha \\
&= \int_{\mathbb{T}} \Im(\frac{\sigma}{A(t)})(-\Re(\partial^4_\alpha z)\cdot\Im(\Lambda(\partial^4_\alpha z)) + \Im(\partial^4_\alpha z)\cdot\Re(\Lambda(\partial^4_\alpha z)))d\alpha
\end{aligned}$$

$$-\int_{\mathbb{T}}\Re(\frac{\sigma}{A(t)})(\Re(\partial_\alpha^4 z)\cdot\Re(\Lambda(\partial_\alpha^4 z))+\Im(\partial_\alpha^4 z)\cdot\Im(\Lambda(\partial_\alpha^4 z)))d\alpha\equiv V_1+V_2,$$

we get

$$\begin{aligned}V_1&=\int_{\mathbb{T}}(-\Lambda(\Im(\frac{\sigma}{A(t)})\Re(\partial_\alpha^4 z))+\Im(\frac{\sigma}{A(t)})\Re(\Lambda(\partial_\alpha^4 z)))\cdot\Im(\partial_\alpha^4 z)d\alpha\\&\leq C\|\frac{\sigma}{A(t)}\|_{\mathcal{C}^{1,\delta}(S)}\|\partial_\alpha^4 z\|^2_{L^2(S)}\leq\exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)}+\|z\|^2_{H^4(S)})\end{aligned}$$

and

$$\begin{aligned}V_2=&-\int_{\mathbb{T}}(\Re(\frac{\sigma}{A(t)})-m(t))(\Re(\partial_\alpha^4 z)\cdot\Re(\Lambda(\partial_\alpha^4 z))+\Im(\partial_\alpha^4 z)\cdot\Im(\Lambda(\partial_\alpha^4 z)))d\alpha\\&-\int_{\mathbb{T}}m(t)(\Re(\partial_\alpha^4 z)\cdot\Re(\Lambda(\partial_\alpha^4 z))+\Im(\partial_\alpha^4 z)\cdot\Im(\Lambda(\partial_\alpha^4 z)))d\alpha\\&\equiv W_1+W_2\end{aligned}$$

where

$$m(t)=\min_\gamma\sigma(\gamma,t).$$

Since $\Re(\frac{\sigma}{A(t)})-m(t)>0$ using $2g\Lambda(g)-\Lambda(g^2)\geq 0$, see [10]

$$\begin{aligned}W_1&\leq\frac{1}{2}\|\Lambda(\Re(\frac{\sigma}{A(t)}))\|_{L^\infty(S)}\|\partial_\alpha^4 z\|^2_{L^2(S)}\leq C\|\frac{\sigma}{A(t)}\|_{\mathcal{C}^{1,\delta}(S)}\|\partial_\alpha^4 z\|^2_{L^2(S)}\\&\leq\exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)}+\|z\|^2_{H^4(S)}),\\W_2&=-m(t)\|\Lambda^{\frac{1}{2}}\partial_\alpha^4 z\|^2_{L^2(S)}\end{aligned}$$

Combining all previous estimates

$$I_7\leq\exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)}+\|z\|^2_{H^4(S)})-m(t)\|\Lambda^{\frac{1}{2}}\partial_\alpha^4 z\|^2_{L^2(S)}.$$

1.1.1.3 Estimates on $\partial_\alpha^4(c(\gamma,t)\cdot\partial_\alpha z(\gamma,t))$ for J_2

In the evolution of the norm of $\partial_\alpha^4 z(\gamma)$ it remains to control the term

$$\begin{aligned}J_2=&\Re\int_{\mathbb{T}}\overline{\partial_\alpha^4 z(\gamma)}\cdot\partial_\alpha^4 c(\gamma)\partial_\alpha z(\gamma)d\alpha+4\Re\int_{\mathbb{T}}\overline{\partial_\alpha^4 z(\gamma)}\cdot\partial_\alpha^3 c(\gamma)\partial_\alpha^2 z(\gamma)d\alpha\\&+6\Re\int_{\mathbb{T}}\overline{\partial_\alpha^4 z(\gamma)}\cdot\partial_\alpha^2 c(\gamma)\partial_\alpha^3 z(\gamma)d\alpha+4\Re\int_{\mathbb{T}}\overline{\partial_\alpha^4 z(\gamma)}\cdot\partial_\alpha c(\gamma)\partial_\alpha^4 z(\gamma)d\alpha\\&+\Re\int_{\mathbb{T}}\overline{\partial_\alpha^4 z(\gamma)}\cdot c(\gamma)\partial_\alpha^5 z(\gamma)d\alpha\equiv I_8+I_9+I_{10}+I_{11}+I_{12}.\end{aligned}$$

Let us recall the formula

$$\begin{aligned}c(\alpha,t)=&\frac{\alpha+\pi}{2\pi}\int_{\mathbb{T}}\frac{\partial_\beta z(\beta,t)}{|\partial_\beta z(\beta,t)|^2}\cdot\partial_\beta BR(z,\varpi)(\beta,t)d\beta\\&-\int_{-\pi}^{\alpha}\frac{\partial_\beta z(\beta,t)}{|\partial_\beta z(\beta,t)|^2}\cdot\partial_\beta BR(z,\varpi)(\beta,t)d\beta,\end{aligned}$$

then

$$
\begin{aligned}
I_9 &= 4\Re \int_{\mathbb{T}} \frac{\overline{\partial_\alpha^4 z(\gamma)}}{A(t)} \cdot \partial_\alpha^2 z(\gamma) \partial_\alpha^3 z(\gamma) \cdot \partial_\alpha BR(z,\varpi)(\gamma) d\alpha \\
&+ 8\Re \int_{\mathbb{T}} \frac{\overline{\partial_\alpha^4 z(\gamma)}}{A(t)} \cdot \partial_\alpha^2 z(\gamma) \partial_\alpha^2 z(\gamma) \cdot \partial_\alpha^2 BR(z,\varpi)(\gamma) d\alpha \\
&+ 4\Re \int_{\mathbb{T}} \frac{\overline{\partial_\alpha^4 z(\gamma)}}{A(t)} \cdot \partial_\alpha^2 z(\gamma) \partial_\alpha z(\gamma) \cdot \partial_\alpha^3 BR(z,\varpi)(\gamma) d\alpha \equiv K_5 + K_6 + K_7
\end{aligned}
$$

and

$$
\begin{aligned}
K_5 &\leq C \|\mathcal{F}(z)\|_{L^\infty(S)} \|z\|_{\mathcal{C}^2(S)} \|z\|_{\mathcal{C}^3(S)} \|BR(z,\varpi)\|_{H^1(S)} \|\partial_\alpha^4 z\|_{L^2(S)}, \\
K_6 &\leq C \|\mathcal{F}(z)\|_{L^\infty(S)} \|z\|_{\mathcal{C}^2(S)}^2 \|\partial_\alpha^2 BR(z,\varpi)\|_{L^2(S)} \|\partial_\alpha^4 z\|_{L^2(S)}, \\
K_7 &\leq C \|\mathcal{F}(z)\|_{L^\infty(S)} \|z\|_{\mathcal{C}^2(S)} \|z\|_{\mathcal{C}^1(S)} \|\partial_\alpha^3 BR(z,\varpi)\|_{L^2(S)} \|\partial_\alpha^4 z\|_{L^2(S)}.
\end{aligned}
$$

Thus

$$I_9 \leq \exp C(\|\mathcal{F}(z)\|_{L^\infty(S)}^2 + \|z\|_{H^4(S)}^2).$$

In the same way,

$$
\begin{aligned}
I_{10} &= -6\Re \int_{\mathbb{T}} \overline{\partial_\alpha^4 z(\gamma)} \cdot \partial_\alpha^3 z(\gamma) \frac{\partial_\alpha^2 z(\gamma)}{|\partial_\alpha z(\gamma)|^2} \cdot \partial_\alpha BR(z,\varpi)(\gamma) d\alpha \\
&- 6\Re \int_{\mathbb{T}} \overline{\partial_\alpha^4 z(\gamma)} \cdot \partial_\alpha^3 z(\gamma) \frac{\partial_\alpha z(\gamma)}{|\partial_\alpha z(\gamma)|^2} \cdot \partial_\alpha^2 BR(z,\varpi)(\gamma) d\alpha \equiv K_8 + K_9
\end{aligned}
$$

where

$$
\begin{aligned}
K_8 &\leq C \|\mathcal{F}(z)\|_{L^\infty(S)} \|z\|_{\mathcal{C}^2(S)} \|z\|_{\mathcal{C}^3(S)} \|\partial_\alpha^4 z\|_{L^2(S)} \|\partial_\alpha BR(z,\varpi)\|_{L^2(S)}, \\
K_9 &\leq C \|\mathcal{F}(z)\|_{L^\infty(S)} \|z\|_{\mathcal{C}^3(S)} \|z\|_{\mathcal{C}^1(S)} \|\partial_\alpha^4 z\|_{L^2(S)} \|\partial_\alpha^2 BR(z,\varpi)\|_{L^2(S)},
\end{aligned}
$$

thus

$$I_{10} \leq \exp C(\|\mathcal{F}(z)\|_{L^\infty(S)}^2 + \|z\|_{H^4(S)}^2).$$

The term I_{11} satisfies

$$
\begin{aligned}
I_{11} &\leq C \|\partial_\alpha c\|_{L^\infty(S)} \|\partial_\alpha^4 z\|_{L^2(S)}^2 \leq C \|\mathcal{F}(z)\|_{L^\infty(S)}^{\frac{1}{2}} \|\partial_\alpha BR\|_{L^\infty(S)} \|\partial_\alpha^4 z\|_{L^2(S)}^2 \\
&\leq \exp C(\|\mathcal{F}(z)\|_{L^\infty(S)}^2 + \|z\|_{H^4(S)}^2)
\end{aligned}
$$

and for I_{12}

$$
\begin{aligned}
I_{12} &= \Re \int_{\mathbb{T}} c(\gamma) \overline{\partial_\alpha^4 z(\gamma)} \cdot \partial_\alpha^5 z(\gamma) d\alpha \\
&= \int_{\mathbb{T}} \Re(c) (\Re(\partial_\alpha^4 z) \Re(\partial_\alpha^5 z) + \Im(\partial_\alpha^4 z) \Im(\partial_\alpha^5 z)) d\alpha \\
&+ \int_{\mathbb{T}} \Im(c) (-\Re(\partial_\alpha^4 z) \Im(\partial_\alpha^5 z) + \Im(\partial_\alpha^4 z) \Re(\partial_\alpha^5 z)) d\alpha \\
&\equiv K_{10} + K_{11}
\end{aligned}
$$

where,

$$K_{10} = -\frac{1}{2}\int_{\mathbb{T}} \Re(\partial_\alpha c)|\partial_\alpha^4 z|^2 d\alpha \leq \|\partial_\alpha z\|_{L^\infty}\|\partial_\alpha^4 z\|^2_{L^2(S)}$$
$$\leq C\|\mathcal{F}(z)\|^{\frac{1}{2}}_{L^\infty(S)}\|\partial_\alpha BR(z,\varpi)\|_{L^\infty(S)}\|\partial_\alpha^4 z\|^2_{L^2(S)} \leq \exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)} + \|z\|^2_{H^4(S)})$$

and

$$K_{11} = \int_{\mathbb{T}} \Im(\partial_\alpha c)\Re(\partial_\alpha^4 z)\Im(\partial_\alpha^4 z)d\alpha + 2\int_{\mathbb{T}} \Im(c)\Im(\partial_\alpha^4 z)\Re(\partial_\alpha^5 z)d\alpha$$
$$\leq \|\partial_\alpha c\|_{L^\infty(S)}\|\partial_\alpha^4 z\|^2_{L^2(S)} - 2\int_{\mathbb{T}} \Im(c)\Im(\partial_\alpha^4 z)\Re(\Lambda(H(\partial_\alpha^4 z)))d\alpha$$
$$\leq \exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)} + \|z\|^2_{H^4(S)}) - 2\int_{\mathbb{T}} \Lambda^{\frac{1}{2}}(\Im(c)\Im(\partial_\alpha^4 z))\Re(\Lambda^{\frac{1}{2}}(H(\partial_\alpha^4 z)))d\alpha$$
$$\leq \exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)} + \|z\|^2_{H^4(S)}) + K\|\Im(c)\|_{H^2(S)}\|\Lambda^{\frac{1}{2}}\partial_\alpha^4 z\|^2_{L^2(S)}.$$

Finally,

$$I_8 = \Re\int_{\mathbb{T}} \frac{\overline{\partial_\alpha^4 z(\gamma)}}{A(t)} \cdot \partial_\alpha z(\gamma)\partial_\alpha^4 z(\gamma) \cdot \partial_\alpha BR(z,\varpi)(\gamma)d\alpha$$
$$- 3\Re\int_{\mathbb{T}} \frac{\overline{\partial_\alpha^4 z(\gamma)}}{A(t)} \cdot \partial_\alpha z(\gamma)\partial_\alpha^3 z(\gamma) \cdot \partial_\alpha^2 BR(z,\varpi)(\gamma)d\alpha$$
$$- 3\Re\int_{\mathbb{T}} \frac{\overline{\partial_\alpha^4 z(\gamma)}}{A(t)} \cdot \partial_\alpha z(\gamma)\partial_\alpha^2 z(\gamma) \cdot \partial_\alpha^3 BR(z,\varpi)(\gamma)d\alpha$$
$$- \Re\int_{\mathbb{T}} \frac{\overline{\partial_\alpha^4 z(\gamma)}}{A(t)} \cdot \partial_\alpha z(\gamma)\partial_\alpha z(\gamma) \cdot \partial_\alpha^4 BR(z,\varpi)(\gamma)d\alpha$$
$$\equiv K_{12} + K_{13} + K_{14} + K_{15}$$

where

$$K_{12} \leq C\|\mathcal{F}(z)\|_{L^\infty(S)}\|z\|_{\mathcal{C}^1(S)}\|\partial_\alpha BR\|_{L^\infty(S)}\|\partial_\alpha^4 z\|^2_{L^2(S)},$$
$$K_{13} \leq C\|\mathcal{F}(z)\|_{L^\infty(S)}\|z\|_{\mathcal{C}^1(S)}\|z\|_{\mathcal{C}^3(S)}\|\partial_\alpha^2 BR\|_{L^2(S)}\|\partial_\alpha^4 z\|_{L^2(S)},$$
$$K_{14} \leq C\|\mathcal{F}(z)\|_{L^\infty(S)}\|z\|_{\mathcal{C}^1(S)}\|z\|_{\mathcal{C}^2(S)}\|\partial_\alpha^3 BR\|_{L^2(S)}\|\partial_\alpha^4 z\|_{L^2(S)}.$$

To estimate K_{15}, we must proceed in the same way we did with J_1. We split $K_{15} = I_3'+I_4'+I_5'+I_6'+I_7'$

$$I_3' = -\Re\int_{\mathbb{T}}\int_{\mathbb{R}} \frac{\overline{\partial_\alpha^4 z(\gamma)}}{A(t)} \cdot \partial_\alpha z(\gamma)\partial_\alpha z(\gamma) \cdot \partial_\alpha^4(\frac{(z(\gamma)-z(\gamma-\beta))^\perp}{|z(\gamma)-z(\gamma-\beta)|^2})\varpi(\gamma-\beta)d\beta d\alpha,$$
$$I_4' = -4\Re\int_{\mathbb{T}}\int_{\mathbb{R}} \frac{\overline{\partial_\alpha^4 z(\gamma)}}{A(t)} \cdot \partial_\alpha z(\gamma)\partial_\alpha z(\gamma) \cdot \partial_\alpha^3(\frac{(z(\gamma)-z(\gamma-\beta))^\perp}{|z(\gamma)-z(\gamma-\beta)|^2})\partial_\alpha\varpi(\gamma-\beta)d\beta d\alpha,$$
$$I_5' = -6\Re\int_{\mathbb{T}}\int_{\mathbb{R}} \frac{\overline{\partial_\alpha^4 z(\gamma)}}{A(t)} \cdot \partial_\alpha z(\gamma)\partial_\alpha z(\gamma) \cdot \partial_\alpha^2(\frac{(z(\gamma)-z(\gamma-\beta))^\perp}{|z(\gamma)-z(\gamma-\beta)|^2})\partial_\alpha^2\varpi(\gamma-\beta)d\beta d\alpha,$$
$$I_6' = -4\Re\int_{\mathbb{T}}\int_{\mathbb{R}} \frac{\overline{\partial_\alpha^4 z(\gamma)}}{A(t)} \cdot \partial_\alpha z(\gamma)\partial_\alpha z(\gamma) \cdot \partial_\alpha(\frac{(z(\gamma)-z(\gamma-\beta))^\perp}{|z(\gamma)-z(\gamma-\beta)|^2})\partial_\alpha^3\varpi(\gamma-\beta)d\beta d\alpha,$$
$$I_7' = -\Re\int_{\mathbb{T}}\int_{\mathbb{R}} \frac{\overline{\partial_\alpha^4 z(\gamma)}}{A(t)} \cdot \partial_\alpha z(\gamma)\partial_\alpha z(\gamma) \cdot (\frac{(z(\gamma)-z(\gamma-\beta))^\perp}{|z(\gamma)-z(\gamma-\beta)|^2})\partial_\alpha^4\varpi(\gamma-\beta)d\beta d\alpha.$$

To study this terms we have to repeat all estimates as in section 1.1.1.1. We select only the terms with different decomposition and we leave to the reader the remainder easy cases. If we consider the term corresponding to M_4 in section 1.1.1.1 we have since

$$\begin{aligned}
\Re(\overline{\partial_\alpha^4 z}\cdot\partial_\alpha z) &= \Re(\partial_\alpha^4 z)\cdot\Re(\partial_\alpha z)+\Im(\partial_\alpha^4 z)\cdot\Im(\partial_\alpha z),\\
\Im(\overline{\partial_\alpha^4 z}\cdot\partial_\alpha z) &= -\Re(\partial_\alpha^4 z)\cdot\Im(\partial_\alpha z)+\Im(\partial_\alpha^4 z)\cdot\Re(\partial_\alpha z),\\
\Re(\partial_\alpha^4 z\cdot\partial_\alpha z) &= \Re(\partial_\alpha^4 z)\cdot\Re(\partial_\alpha z)-\Im(\partial_\alpha^4 z)\cdot\Im(\partial_\alpha z),\\
\Im(\partial_\alpha^4 z\cdot\partial_\alpha z) &= \Im(\partial_\alpha^4 z)\cdot\Re(\partial_\alpha z)+\Re(\partial_\alpha^4 z)\cdot\Im(\partial_\alpha z),
\end{aligned}$$

and

$$\partial_\alpha z\cdot\partial_\alpha^4 z=-3\partial_\alpha^2 z\cdot\partial_\alpha^3 z.$$

we can write

$$\begin{aligned}
\Re(\overline{\partial_\alpha^4 z}\cdot\partial_\alpha z) &= \Re(-3\partial_\alpha^2 z\cdot\partial_\alpha^3 z)+2\Im(\partial_\alpha^4 z)\cdot\Im(\partial_\alpha z),\\
\Im(\overline{\partial_\alpha^4 z}\cdot\partial_\alpha z) &= \Im(-3\partial_\alpha^2 z\cdot\partial_\alpha^3 z)-2\Re(\partial_\alpha^4 z)\cdot\Im(\partial_\alpha z).
\end{aligned}$$

Thus

$$\begin{aligned}
M_4' &= -2\pi\Re\int_{\mathbb{T}}\frac{\overline{\partial_\alpha^4 z(\gamma)}}{A^2(t)}\cdot\partial_\alpha z(\gamma)\partial_\alpha z(\gamma)\cdot\Lambda(\partial_\alpha^4 z^\perp)(\gamma)\varpi(\gamma)d\alpha\\
&= -2\pi\int_{\mathbb{T}}\Re(\overline{\partial_\alpha^4 z}\cdot\partial_\alpha z)\Re(\partial_\alpha z\cdot\Lambda(\partial_\alpha^4 z^\perp)\frac{\varpi}{A^2(t)})-\Im(\overline{\partial_\alpha^4 z}\cdot\partial_\alpha z)\Im(\partial_\alpha z\cdot\Lambda(\partial_\alpha^4 z^\perp)\frac{\varpi}{A^2(t)})d\alpha\\
&= -2\pi\int_{\mathbb{T}}(\Re(-3\partial_\alpha^2 z\cdot\partial_\alpha^3 z)+2\Im(\partial_\alpha^4 z)\cdot\Im(\partial_\alpha z))\Re(\partial_\alpha z\cdot\Lambda(\partial_\alpha^4 z^\perp)\frac{\varpi}{A^2(t)})d\alpha\\
&+ 2\pi\int_{\mathbb{T}}(\Im(-3\partial_\alpha^2 z\cdot\partial_\alpha^3 z)-2\Re(\partial_\alpha^4 z)\cdot\Im(\partial_\alpha z))\Im(\partial_\alpha z\cdot\Lambda(\partial_\alpha^4 z^\perp)\frac{\varpi}{A^2(t)})d\alpha\\
&\equiv Q_4'^{1}+N_6'
\end{aligned}$$

we have,

$$\begin{aligned}
N_5' &= -2\pi\int_{\mathbb{T}}\Re(-3\partial_\alpha^2 z\cdot\partial_\alpha^3 z)\Re(\partial_\alpha z\cdot\Lambda(\partial_\alpha^4 z^\perp)\frac{\varpi}{A^2(t)})d\alpha\\
&- 4\pi\int_{\mathbb{T}}\Im(\partial_\alpha^4 z)\cdot\Im(\partial_\alpha z)\Re(\partial_\alpha z\cdot\Lambda(\partial_\alpha^4 z^\perp)\frac{\varpi}{A^2(t)})d\alpha\\
&\equiv O_3'+O_4'.
\end{aligned}$$

Clearly,

$$O_3'\le\exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)}+\|z\|^2_{H^4(S)}).$$

Since

$$\Re(\partial_\alpha z\cdot\Lambda(\partial_\alpha^4 z^\perp)\frac{\varpi}{A^2(t)})=\Re(\partial_\alpha z\frac{\varpi}{A^2(t)})\cdot\Re(\Lambda(\partial_\alpha^4 z^\perp))-\Im(\partial_\alpha z\frac{\varpi}{A^2(t)})\cdot\Im(\Lambda(\partial_\alpha^4 z^\perp))$$

we take,

$$
\begin{aligned}
O_4' &= -4\pi\int_{\mathbb{T}} \Im(\partial_\alpha^4 z)\cdot\Im(\partial_\alpha z)\Re(\partial_\alpha z\frac{\varpi}{A^2(t)})\cdot\Re(\Lambda(\partial_\alpha^4 z^\perp))d\alpha\\
&+4\pi\int_{\mathbb{T}} \Im(\partial_\alpha^4 z)\cdot\Im(\partial_\alpha z)\Im(\partial_\alpha z\frac{\varpi}{A^2(t)})\cdot\Im(\Lambda(\partial_\alpha^4 z^\perp))d\alpha\\
&\leq \exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)}+\|z\|^2_{H^4(S)})+k\|\Im(\partial_\alpha z)\Re(\partial_\alpha z\frac{\varpi}{A^2(t)})\|_{H^2(S)}\|\Lambda^{\frac{1}{2}}\partial_\alpha^4 z\|^2_{L^2(S)}\\
&+\exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)}+\|z\|^2_{H^4(S)})+c\|\Im(\partial_\alpha z)\Im(\partial_\alpha z\frac{\varpi}{A^2(t)})\|_{H^2(S)}\|\Lambda^{\frac{1}{2}}\partial_\alpha^4 z\|^2_{L^2(S)}.
\end{aligned}
$$

For N_6'

$$
\begin{aligned}
N_6' &= 2\pi\int_{\mathbb{T}} \Im(-3\partial_\alpha^2 z\cdot\partial_\alpha^3 z)\Im(\partial_\alpha z\cdot\Lambda(\partial_\alpha^4 z^\perp)\frac{\varpi}{A^2(t)})d\alpha\\
&-4\pi\int_{\mathbb{T}} \Re(\partial_\alpha^4 z)\cdot\Im(\partial_\alpha z)\Im(\partial_\alpha z\cdot\Lambda(\partial_\alpha^4 z^\perp)\frac{\varpi}{A^2(t)})d\alpha\\
&\equiv O_5'+O_6'
\end{aligned}
$$

Clearly,

$$
O_5' \leq \exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)}+\|z\|^2_{H^4(S)}).
$$

Since,

$$
\Im(\partial_\alpha z\cdot\Lambda(\partial_\alpha^4 z^\perp)\frac{\varpi}{A^2(t)}) = \Re(\partial_\alpha z\frac{\varpi}{A^2(t)})\cdot\Im(\Lambda(\partial_\alpha^4 z^\perp))+\Im(\partial_\alpha z\frac{\varpi}{A^2(t)})\cdot\Re(\Lambda(\partial_\alpha^4 z^\perp))
$$

we have,

$$
\begin{aligned}
O_6' &= -4\pi\int_{\mathbb{T}} \Re(\partial_\alpha^4 z)\cdot\Im(\partial_\alpha z)\Re(\partial_\alpha z\frac{\varpi}{A^2(t)})\cdot\Im(\Lambda(\partial_\alpha^4 z^\perp))d\alpha\\
&-4\pi\int_{\mathbb{T}} \Re(\partial_\alpha^4 z)\cdot\Im(\partial_\alpha z)\Im(\partial_\alpha z\frac{\varpi}{A^2(t)})\cdot\Re(\Lambda(\partial_\alpha^4 z^\perp))d\alpha\\
&\leq \exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)}+\|z\|^2_{H^4(S)})+k\|\Im(\partial_\alpha z)\Re(\partial_\alpha z\frac{\varpi}{A^2(t)})\|_{H^2(S)}\|\Lambda^{\frac{1}{2}}\partial_\alpha^4 z\|^2_{L^2(S)}\\
&+\exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)}+\|z\|^2_{H^4(S)})+c\|\Im(\partial_\alpha z)\Im(\partial_\alpha z\frac{\varpi}{A^2(t)})\|_{H^2(S)}\|\Lambda^{\frac{1}{2}}\partial_\alpha^4 z\|^2_{L^2(S)}.
\end{aligned}
$$

Using a similar method for the rest of non-integrable terms we obtain

$$
\begin{aligned}
J_2 &\leq \exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)}+\|z\|^2_{H^4(S)})\\
&+C(\|\Im(\partial_\alpha z)\Re(\partial_\alpha z\frac{\varpi}{A^2(t)})\|_{H^2(S)}+\|\Im(\partial_\alpha z)\Im(\partial_\alpha z\frac{\varpi}{A^2(t)})\|_{H^2(S)}+\|\Im(c)\|_{H^2(S)})\|\Lambda^{\frac{1}{2}}\partial_\alpha^4 z\|^2_{L^2(S)}.
\end{aligned}
$$

In conclusion,

$$
\begin{aligned}
I_1 &\leq \exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)}+\|z\|^2_{H^4(S)})+C[\|\Im(\frac{\varpi}{A(t)})\|_{H^2(S)}\|\Im(\partial_\alpha z)\Re(\partial_\alpha z\frac{\varpi}{A^2(t)})\|_{H^2(S)}\\
&+\|\Im(\partial_\alpha z)\Im(\partial_\alpha z\frac{\varpi}{A^2(t)})\|_{H^2(S)}+\|\Im(c)\|_{H^2(S)}-m(t)]\|\Lambda^{\frac{1}{2}}\partial_\alpha^4 z\|^2_{L^2(S)}
\end{aligned}
$$

and therefore

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}}|\partial_\alpha^4 z(\gamma)|^2 d\alpha = I_1 + I_2\\&\leq \exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)} + \|z\|^2_{H^4(S)}) + C[\|\Im(\frac{\varpi}{A(t)})\|_{H^2(S)}\|\Im(\partial_\alpha z)\Re(\partial_\alpha z\frac{\varpi}{A^2(t)})\|_{H^2(S)}\\&+ \|\Im(\partial_\alpha z)\Im(\partial_\alpha z\frac{\varpi}{A^2(t)})\|_{H^2(S)} + \|\Im(c)\|_{H^2(S)} - m(t) + 2\lambda]\|\Lambda^{\frac{1}{2}}\partial_\alpha^4 z\|^2_{L^2(S)}.\end{aligned} \tag{1.10}$$

1.2 The evolution of the minimum of $\sigma(\gamma, t)$

For the one-phase Muskat problem the Rayleigh-taylor condition is:

$$\sigma(\alpha, t) = -\nabla p(z(\alpha, t), t)\cdot \partial_\alpha^\perp z(\alpha, t).$$

where p is the pressure in the domain Ω. Taking the divergence in Darcy's law we obtain

$$\Delta p(x, t) = 0,$$

for any $x \in \Omega(t)$. Since we consider velocities with mean zero vorticity, we have $u \in L^2(\Omega)$ and finite energy settings. Then,

$$\lim_{x_2\to-\infty} u(x, t) = 0,$$

and therefore Darcy's law gives

$$\begin{aligned}\lim_{x_2\to-\infty} \partial_{x_1} p(x, t) &= 0,\\ \lim_{x_2\to-\infty} \partial_{x_2} p(x, t) &= g\rho^2.\end{aligned}$$

In conclusion, $p(x, t) \to +\infty$ when $x_2 \to -\infty$ and the minimum of p in Ω is attained on the free boundary $z(\alpha, t)$. Therefore, using Hopf's lemma,

$$\begin{cases}\Delta p = 0\\ \exists x^* \in \partial\Omega / p(x^*, t) < p(x, t)\end{cases} \Rightarrow \nabla p(x^*, t)\cdot \partial_\alpha^\perp z(\alpha, t) < 0$$

Thus, $\sigma(\alpha, t) > 0$.

In spite of this property, we need to get an a priori estimate for the evolution of the minimum of σ in the strip S in order to absorb the high order terms in (1.10).

Recall that

$$\sigma(\alpha, t) = \frac{\mu^2}{\kappa} BR(z, \varpi)(\alpha, t)\cdot \partial_\alpha^\perp z(\alpha, t) + g\rho^2 \partial_\alpha z_1(\alpha, t). \tag{1.11}$$

Lemma 1.2.1. *Let $z(\gamma, t)$ be a solution of the system with $z(\gamma, t) \in \mathcal{C}([0, T]; H^4(S)) \cap \mathcal{C}^1([0, T]; H^3(S))$, and*

$$m(t) = \min_\gamma \sigma(\gamma, t).$$

Then

$$m(t) \geq m(0) - \int_0^t \exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)} + \|z\|^2_{H^4(S)}) ds.$$

Proof. We may consider $\gamma_t\in\mathbb{C}$ such that

$$m(t)=\min_{\gamma}\sigma(\gamma,t)=\sigma(\gamma_t,t).$$

We may calculate the derivative of $m(t)$, to obtain

$$m'(t)=\sigma_t(\gamma_t,t).$$

The identity (1.11) yields,

$$\begin{aligned}\sigma_t(\gamma,t)=&\frac{\mu^2}{\kappa}\partial_t BR(z,\varpi)(\gamma,t)\cdot\partial_\alpha^\perp z(\gamma,t)+i\lambda\frac{\mu^2}{\kappa}\partial_\alpha BR(z,\varpi)(\gamma,t)\cdot\partial_\alpha^\perp z(\gamma,t)\\&+\frac{\mu^2}{\kappa}BR(z,\varpi)(\gamma,t)\cdot\partial_\alpha^\perp z_t(\gamma,t)+\frac{\mu^2}{\kappa}BR(z,\varpi)(\gamma,t)\cdot i\lambda\partial_\alpha^2 z(\gamma,t)\\&+g\rho^2\partial_\alpha z_{1t}(\gamma,t)+g\rho^2\partial_\alpha^2 z_1(\gamma,t)\equiv R_1+R_2+R_3+R_4+R_5+R_6.\end{aligned}$$

We can easily estimate,

$$\begin{aligned}|R_2|&\le\lambda\frac{\mu^2}{\kappa}\|\partial_\alpha BR(z,\varpi)\|_{L^\infty(S)}\|\partial_\alpha z\|_{L^\infty(S)}\le\exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)}+\|z\|^2_{H^4(S)}),\\|R_4|&\le\lambda\frac{\mu^2}{\kappa}\|BR(z,\varpi)\|_{L^\infty(S)}\|z\|_{\mathcal{C}^2(S)}\le\exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)}+\|z\|^2_{H^4(S)}),\\|R_6|&\le g\rho^2\|z\|_{\mathcal{C}^2(S)}\le\exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)}+\|z\|^2_{H^4(S)}),\end{aligned}$$

and we have,

$$|R_3|+|R_5|\le C(\|BR(z,\varpi)\|_{L^\infty(S)}+1)\|\partial_\alpha z_t\|_{L^\infty(S)}.$$

Since $z_t(\gamma)=BR(z,\varpi)(\gamma)+c(\gamma)\partial_\alpha z(\gamma)$,

$$\begin{aligned}\|\partial_\alpha z_t\|_{L^\infty(S)}&\le\|\partial_\alpha BR(z,\varpi)\|_{L^\infty(S)}+\|\partial_\alpha c\|_{L^\infty(S)}\|\partial_\alpha z\|_{L^\infty(S)}+\|c\|_{L^\infty(S)}\|\partial_\alpha^2 z\|_{L^\infty(S)}\\&\le C\|\partial_\alpha BR(z,\varpi)\|_{L^\infty(S)}(1+\|\mathcal{F}(z)\|^{\frac12}_{L^\infty(S)}\|z\|_{\mathcal{C}^2(S)})\le\exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)}+\|z\|^2_{H^4(S)}).\end{aligned}$$

Then,

$$|R_3+R_5|\le\exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)}+\|z\|^2_{H^4(S)}).$$

Recall that

$$BR(z,\varpi)(\gamma)=\frac{1}{2\pi}\int_{\mathbb{T}}\frac{\Delta z^\perp}{|\Delta z|^2}\varpi(\gamma-\beta)d\beta,$$

then

$$\begin{aligned}BR_t(z,\varpi)(\gamma)=&\frac{1}{2\pi}\int_{\mathbb{T}}\frac{\Delta z_t^\perp}{|\Delta z|^2}\varpi(\gamma-\beta)d\beta-\frac{1}{\pi}\int_{\mathbb{T}}\frac{\Delta z^\perp(\Delta z\cdot\Delta z_t)}{|\Delta z|^4}\varpi(\gamma-\beta)d\beta\\&+\frac{1}{2\pi}\int_{\mathbb{T}}\frac{\Delta z^\perp}{|\Delta z|^2}\varpi_t(\gamma-\beta)d\beta\equiv J_1+J_2+J_3.\end{aligned}$$

We get

$$J_1=\frac{1}{2\pi}\int_{\mathbb{T}}\Delta z_t^\perp\varpi(\gamma-\beta)(\frac{1}{|\Delta z|^2}-\frac{1}{A(t)\beta^2})d\beta$$

$$+\frac{1}{2\pi}\int_{\mathbb{T}}\frac{\Delta z_t^{\perp}}{A(t)\beta^2}\varpi(\gamma-\beta)d\beta \equiv K_1+K_2.$$

Using that $\Delta z_t^{\perp}=\beta\int_0^1\partial_\alpha z_t(\phi)ds$,

$$\begin{aligned}K_1&=\frac{1}{2\pi}\int_{\mathbb{T}}\varpi(\gamma-\beta)\beta\int_0^1\partial_\alpha z_t^{\perp}(\phi)dsB(\gamma,\beta)d\beta\\&\le C\|\mathcal{F}(z)\|^{\frac{3}{2}}_{L^\infty(S)}\|z\|_{\mathcal{C}^2(S)}\|\varpi\|_{L^\infty(S)}\|\partial_\alpha z_t\|_{L^\infty(S)}\le \exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)}+\|z\|^2_{H^4(S)}).\end{aligned}$$

Since

$$\partial_\alpha^2 z_t=\partial^2 BR(z,\varpi)+\partial_\alpha^2 c\partial_\alpha z+2\partial_\alpha c\partial_\alpha^2 z+c\partial^3 z$$

and

$$\begin{aligned}&\|\partial_\alpha^2 BR(z,\varpi)\|_{L^\infty(S)}\le \exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)}+\|z\|^2_{H^4(S)}),\\&\|\partial_\alpha^2 c\partial_\alpha z\|_{L^\infty(S)}=\|\frac{\partial_\alpha^2 z}{|\partial_\alpha z|^2}\cdot\partial_\alpha BR(z,\varpi)\partial_\alpha z\|_{L^\infty(S)}+\|\frac{\partial_\alpha z}{|\partial_\alpha z|^2}\cdot\partial_\alpha^2 BR(z,\varpi)\partial_\alpha z\|_{L^\infty(S)}\\&\le C\|\mathcal{F}(z)\|^{\frac{1}{2}}_{L^\infty(S)}\|z\|_{\mathcal{C}^2(S)}(\|\partial_\alpha BR(z,\varpi)\|_{L^\infty(S)}+\|\partial_\alpha^2 BR(z,\varpi)\|_{L^\infty(S)})\\&\le \exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)}+\|z\|^2_{H^4(S)}),\\&2\|\partial_\alpha c\partial_\alpha^2 z\|_{L^\infty(S)}\le 4\|\mathcal{F}(z)\|^{\frac{1}{2}}_{L^\infty(S)}\|\partial_\alpha BR(z,\varpi)\|_{L^\infty(S)}\|\partial_\alpha^2 z\|_{L^\infty(S)}\\&\le \exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)}+\|z\|^2_{H^4(S)}),\end{aligned}$$

then

$$\|\partial_\alpha^2 z_t\|_{L^\infty(S)}\le \exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)}+\|z\|^2_{H^4(S)}).$$

Thus,

$$\begin{aligned}K_2&=\frac{1}{2\pi}\int_{\mathbb{T}}\frac{\int_0^1\partial_\alpha z_t^{\perp}(\phi)ds}{A(t)\beta}\varpi(\gamma-\beta)d\beta\\&=\frac{1}{2\pi}\int_{\mathbb{T}}\frac{\int_0^1\partial_\alpha z_t^{\perp}(\phi)-\partial_\alpha z_t^{\perp}(\gamma)ds}{A(t)\beta}\varpi(\gamma-\beta)d\beta+\frac{1}{2\pi}\int_{\mathbb{T}}\frac{\partial_\alpha z_t^{\perp}(\gamma)}{A(t)\beta}\varpi(\gamma-\beta)d\beta\\&=\frac{1}{2\pi}\int_{\mathbb{T}}\frac{\int_0^1\int_0^1\partial_\alpha^2 z_t^{\perp}(\psi)(s-1)dtds}{A(t)}\varpi(\gamma-\beta)d\beta+\frac{1}{2}\frac{\partial_\alpha z_t^{\perp}(\gamma)}{A(t)}H(\varpi)(\gamma)\\&\le C\|\mathcal{F}(z)\|_{L^\infty(S)}\|\partial_\alpha^2 z_t\|_{L^\infty(S)}\|\varpi\|_{L^\infty(S)}+K\|\mathcal{F}(z)\|_{L^\infty(S)}\|\partial_\alpha z_t\|_{L^\infty(S)}\|\varpi\|_{\mathcal{C}^\delta(S)}.\end{aligned}$$

Therefore,

$$J_1\le \exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)}+\|z\|^2_{H^4(S)}).$$

In the same way, it is easy to see that

$$J_2\le \exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)}+\|z\|^2_{H^4(S)}).$$

Finally, since

$$\frac{\Delta z^{\perp}}{|\Delta z|^2}-\frac{\partial_\alpha z^{\perp}(\gamma)}{A(t)\beta}=\frac{\beta^2\int_0^1\int_0^1\partial_\alpha^2 z(\psi)(s-1)dtds}{|\Delta z|^2}$$

$$+\frac{\beta^2\partial_\alpha z(\gamma)\int_0^1\int_0^1\partial_\alpha^2 z(\psi)(s-1)dtds\cdot\int_0^1[\partial_\alpha z(\gamma)+\partial_\alpha z(\phi)]ds}{A(t)|\Delta z|^2},$$

$$\begin{aligned}J_3 &= \frac{1}{2\pi}\int_{\mathbb{T}}(\frac{\Delta z^\perp}{|\Delta z|^2}-\frac{\partial_\alpha z^\perp(\gamma)}{A(t)\beta})\varpi_t(\gamma-\beta)d\beta+\frac{1}{2\pi}\int_{\mathbb{T}}\frac{\partial_\alpha z^\perp(\gamma)}{A(t)\beta}\varpi_t(\gamma-\beta)d\beta\\ &\equiv K_5+K_6\end{aligned}$$

where

$$\begin{aligned}K_5 &\le C\|\mathcal{F}(z)\|_{L^\infty(S)}\|z\|_{\mathcal{C}^2(S)}\|\varpi_t\|_{L^2(S)},\\ K_6 &= \frac{1}{2}\frac{\partial_\alpha z^\perp(\gamma)}{A(t)}H(\varpi_t)(\gamma)\le C\|\mathcal{F}(z)\|^{\frac{1}{2}}_{L^\infty(S)}\|\varpi_t\|_{\mathcal{C}^\delta(S)}.\end{aligned}$$

In order to control $\|\varpi_t\|_{\mathcal{C}^\delta(S)}$ we proceed as in section 9 in [11].

Therefore,

$$|\sigma_t(\gamma,t)|\le \exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)}+\|z\|^2_{H^4(S)})$$

given us,

$$m'(t)\ge -\exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)}+\|z\|^2_{H^4(S)})$$

for almost every t. And a further integration yields

$$m(t)\ge m(0)-\int_0^t \exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)}+\|z\|^2_{H^4(S)})ds.$$

■

1.3 Instant Analyticity

Theorem 1.3.1. *Let $z(\alpha,0)=z_0(\alpha)\in H^4$, $\mathcal{F}(z_0)(\alpha,\beta)\in L^\infty$. Then there exists a solution of the Muskat problem $z(\alpha,t)$ defined for $0<t\le T$ that continues analytically into the strip $S(t)=\{\alpha\pm i\varsigma : |\varsigma|<\lambda t\}$ for each t. Here, λ and T are determined by upper bounds of the H^4 norm and the arc-chord constant of the initial data and a positive lower bound of the $\sigma(\alpha,0)$. Moreover, for $0<t\le T$, the quantity*

$$\sum_{\pm}\int_{\mathbb{T}}(|z(\alpha\pm i\lambda t)-(\alpha\pm i\lambda t)|^2+|\partial_\alpha^4 z(\alpha\pm i\lambda t)|^2)d\alpha$$

is bounded by a constant determinate by upper bounds for the H^4 norm and the arc-chord constant of the initial data and a positive lower bound of $\sigma(\alpha,0)$.

Proof. For all estimates in above sections we have finally

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}}|\partial_\alpha^4 z(\alpha\pm i\lambda t)|^2 &\le \exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)}+\|z\|^2_{H^4(S)})\\ &\quad+(2\lambda+C\|f\|(t)-m(t))\|\Lambda^{\frac{1}{2}}\partial_\alpha^4 z\|^2_{L^2(S)}(t)\end{aligned}$$

where

$$\|f\|(t) = \|\Im(\frac{\varpi}{A(t)})\|_{H^2(S)} + \|\Im(\partial_\alpha z)\Re(\partial_\alpha z\varpi)\|_{H^2(S)} + \|\Im(\partial_\alpha z)\Im(\partial_\alpha z\varpi)\|_{H^2(S)} + \|\Im(c)\|_{H^2(S)}.$$

Note that $\|f\|(0) = 0$. If $2\lambda - m(0) < 0$ we will show that

$$2\lambda + K\|f\|(t) - m(t) < 0$$

for short time. It yields

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}} |\partial_\alpha^4 z(\alpha \pm i\lambda t)|^2 d\alpha \leq \exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)} + \|z\|^2_{H^4(S)})$$

as long as $2\lambda + K\|f\|(t) - m(t) < 0$. We proceed as in section 8 in [11] to show that

$$\frac{d}{dt}\|\mathcal{F}(z)\|_{L^\infty(S)} \leq \exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)} + \|z\|^2_{H^4(S)}).$$

From the two inequalities above and (1.5) it is easy to obtain a priori energy estimates that depend upon the negativity of $2\lambda + K\|f\|(t) - m(t)$. We denote

$$\|z\|_{RT}(t) \equiv \|\mathcal{F}(z)\|^2_{L^\infty(S)}(t) + \|z\|^2_{L^2(S)} + \frac{1}{m(t) - 2\lambda - C\|f\|}.$$

At this point it is easy to find

$$\|f\| \leq \exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)} + \|z\|^2_{H^4(S)})$$

and

$$\frac{d}{dt}(\frac{1}{m(t) - 2\lambda - C\|f\|}) \leq \frac{\exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)} + \|z\|^2_{H^4(S)})}{(m(t) - 2\lambda - C\|f\|)^2}$$

then,

$$\frac{d}{dt}\|z\|_{RT}(t) \leq \exp C(\|z\|_{RT}(t))$$

and therefore,

$$\|z\|_{RT} \leq -\log(\exp(-C\|z\|_{RT}(0) - C^2 t)).$$

Now we approximate the problem as follows,

$$\begin{cases} z^\epsilon_t(\alpha, t) = BR(z^\epsilon, \varpi^\epsilon)(\alpha, t) + c^\epsilon(\alpha, t)\partial_\alpha z^\epsilon(\alpha, t) \\ z^\epsilon(\alpha, 0) = \phi_\epsilon * z_0(\alpha) \end{cases}$$

where

$$\begin{aligned} c^\epsilon(\alpha, t) = & \frac{\alpha + \pi}{2\pi}\int_{\mathbb{T}} \frac{\partial_\alpha z^\epsilon(\alpha, t)}{|\partial_\alpha z^\epsilon(\alpha, t)|^2} \cdot \partial_\alpha BR(z^\epsilon, \varpi^\epsilon)(\alpha, t) d\alpha \\ & - \int_{-\pi}^{\alpha} \frac{\partial_\alpha z^\epsilon(\beta, t)}{|\partial_\alpha z^\epsilon(\beta, t)|^2} \cdot \partial_\alpha BR(z^\epsilon, \varpi^\epsilon)(\beta, t) d\beta, \end{aligned}$$

$$\varpi^\epsilon(\alpha, t) = -\phi_\epsilon * \phi_\epsilon * (2BR(z^\epsilon, \varpi^\epsilon) \cdot \partial_\alpha z^\epsilon)(\alpha) - 2\kappa\frac{\rho^2}{\mu^2}\phi_\epsilon * \phi_\epsilon * (\partial_\alpha z_2{}^\epsilon)(\alpha)$$

where $\phi_\epsilon(\alpha) = \phi(\frac{\alpha}{\epsilon})/\epsilon$ for $\epsilon > 0$ and ϕ the heat kernel.

Picard's Theorem yields the existence of a solution $z^\epsilon(\alpha)$ in $\mathcal{C}([0, T^\epsilon); H^4)$ which is analytic in the whole space for z_0 satisfying the arc-chord condition and ϵ small enough. Using the same techniques we have devoted above we obtain a bound for $z^\epsilon(\alpha, t)$ in H^4 in the strip $S(t)$ for a small enough T which is independent of ϵ. We need arc-chord condition, $z_0 \in H^4$ and $2\lambda - m(0) < 0$. Then we pass to the limit. ■

1.4 Decay estimates on the strip of analyticity

Theorem 1.4.1. *Let $z(\alpha, 0) = z^0(\alpha)$ be an analytic curve in the strip*

$$S = \{\alpha + i\varsigma \in \mathbb{C} : |\varsigma| < h(0)\},$$

with $h(0) > 0$ and satisfying:

* *The arc-chord condition, $\mathcal{F}(z^0)(\alpha + i\varsigma, \beta) \in L^\infty(S \times \mathbb{R})$*
* *The curve $z^0(\alpha)$ is real for real α*
* *The functions $z_1^0(\alpha) - \alpha$ and ${z_2}^0(\alpha)$ are periodic with period 2π*
* *The functions ${z_1}^0(\alpha) - \alpha$ and ${z_2}^0(\alpha)$ belong to $H^4(S)$*

Then there exist a time T and a solution of the Muskat problem $z(\alpha, t)$ defined for $0 < t \leq T$ that continues analytically into some complex strip for each fixed $t \in [0, T]$. Here T is either a small constant depending only on $\exp C(\|\mathcal{F}(z^0)\|^2_{L^\infty(S)} + \|z^0\|^2_{L^2(S)})$.

We will use the following:

Lemma 1.4.1. *Let $\psi(\alpha \pm i\xi) = \sum_{k=-N}^{N} A_k(t) e^{ik\alpha} e^{\pm k\xi}$ and $h(t) > 0$ be a decreasing function of t. Then*

$$\begin{aligned}
\frac{\partial}{\partial t} \sum_{\pm} \int_{\mathbb{T}} |\psi(\alpha \pm ih(t))|^2 d\alpha \leq{}& \frac{h'(t)}{10} \sum_{\pm} \int_{\mathbb{T}} \Lambda\psi(\alpha \pm ih(t)) \overline{\psi(\alpha \pm ih(t))} d\alpha \\
&- 10h'(t) \int_{\mathbb{T}} \Lambda\psi(\alpha) \overline{\psi}(\alpha) d\alpha + 2\Re \sum_{\pm} \int_{\mathbb{T}} \psi_t(\alpha \pm ih(t)) \overline{\psi(\alpha \pm ih(t))} d\alpha.
\end{aligned}$$

This lemma is a corollary of the lemma 4.2 in [5] and it allows us to prove the Theorem 1.4.1.

Proof of Theorem 1.4.1. The norms $\|z\|_{L^2(S)}$ and $\|z\|_{H^k(S)}$ are defined as before using the new strip $S(t)$ defined by

$$S(t) = \{\alpha + i\varsigma \in \mathbb{C} : |\varsigma| < h(t)\}$$

where $h(t)$ is a positive decreasing function of t.

We use the Galerkin approximation of equation $z_t(\alpha, t) = BR(z, \varpi)\alpha, t) + c(\alpha, t)\partial_\alpha z(\alpha, t)$, i.e.

$$z_t^{[N]}(\gamma, t) = \Pi_N[J[z^{[N]}]](\gamma, t)$$

where $\gamma \in \overline{S(t)}$, Π_N will be defined below, and

$$J[z](\alpha, t) = BR(z, \varpi)(\alpha, t) + c(\alpha, t)\partial_\alpha z(\alpha, t).$$

We impose the initial condition

$$z^{[N]}(\alpha,0)=z^{[N]}(\alpha).$$

Here, for a large enough positive integer N, we define $z^{[N]}(\alpha,0)$ from $z^0(\alpha)$ by using the projection

$$\Pi_N:\sum_{-\infty}^{\infty}A_k e^{ik\alpha}\to\sum_{-N}^{N}A_k e^{ik\alpha}.$$

We defined $z^{[N]}(\alpha)$ by stipulating that

$$z_1{}^{[N]}(\alpha)-\alpha=\Pi_N[z_1{}^0(\alpha)-\alpha]$$

and

$$z_2{}^{[N]}(\alpha)=\Pi_N[z_2{}^0(\alpha)].$$

For N large enough, the functions $z^{[N]}(\alpha,0)$ satisfy the arc-chord and Rayleigh-Taylor conditions.

We shall consider the evolution of the most singular quantity

$$\sum_{\pm}\int_{\mathbb{T}}|\partial_\alpha^4 z^{[N]}(\alpha\pm ih_N(t),t)|^2 d\alpha$$

where $h_N(t)$ is a smooth positive decreasing function on t, with $h_N(0)=h(0)$, which will be given below. Also we denote

$$S_N(t)=\{\alpha+i\varsigma\in\mathbb{C}:|\varsigma|<h_N(t)\}.$$

We will drop the dependency on N from $z^{[N]}$ and $h_N(t)$ in our notation. Using lemma above,

$$\begin{aligned}\frac{d}{dt}\sum_{\pm}\int_{\mathbb{T}}|\partial_\alpha^4 z_j(\alpha\pm ih(t))|^2 d\alpha &\leq \frac{h'(t)}{10}\sum_{\pm}\int_{\mathbb{T}}\Lambda(\partial_\alpha^4 z_j)(\alpha\pm ih(t))\overline{\partial_\alpha^4 z_j(\alpha\pm ih(t))}\\ &-10h'(t)\int_{\mathbb{T}}\Lambda(\partial_\alpha^4 z_j)(\alpha)\overline{\partial_\alpha^4 z_j(\alpha)}d\alpha+2\sum_{\pm}\Re\int_{\mathbb{T}}\Pi_N[\partial_\alpha^4 J_j[z]](\alpha\pm ih(t))\overline{\partial_\alpha^4 z_j(\alpha\pm ih(t))}.\end{aligned}$$

Since $\partial_\alpha^4 z_j(\alpha\pm ih(t))$ is a trigonometric polynomial in the range of Π_N

$$\begin{aligned}&2\sum_{\pm}\Re\int_{\mathbb{T}}\Pi_N[\partial_\alpha^4 J_j[z]](\alpha\pm ih(t))\overline{\partial_\alpha^4 z_j(\alpha\pm ih(t))}\\ &=2\sum_{\pm}\Re\int_{\mathbb{T}}\partial_\alpha^4 J_j[z](\alpha\pm ih(t))\Pi_N[\overline{\partial_\alpha^4 z_j(\alpha\pm ih(t))}]\\ &=2\sum_{\pm}\Re\int_{\mathbb{T}}\partial_\alpha^4 J_j[z](\alpha\pm ih(t))\overline{\partial_\alpha^4 z_j(\alpha\pm ih(t))}\end{aligned}$$

then,

$$\frac{d}{dt}\sum_{\pm}\int_{\mathbb{T}}|\partial_\alpha^4 z_j(\alpha\pm ih(t))|^2 d\alpha\leq\frac{h'(t)}{10}\sum_{\pm}\int_{\mathbb{T}}\Lambda(\partial_\alpha^4 z_j)(\alpha\pm ih(t))\overline{\partial_\alpha^4 z_j(\alpha\pm ih(t))}$$

$$
\begin{aligned}
&-10h'(t)\int_{\mathbb{T}}\Lambda(\partial_\alpha^4 z_j)(\alpha)\overline{\partial_\alpha^4 z_j(\alpha)}d\alpha+2\sum_{\pm}\Re\int_{\mathbb{T}}\partial_\alpha^4 BR(z_j,\varpi)(\alpha\pm ih(t))\overline{\partial_\alpha^4 z_j(\alpha\pm ih(t))}\\
&+2\sum_{\pm}\Re\int_{\mathbb{T}}\partial_\alpha^4(c(\alpha\pm ih(t))\partial_\alpha z_j(\alpha\pm ih(t)))\overline{\partial_\alpha^4 z_j(\alpha\pm ih(t))}\\
&\equiv M_1+M_2+M_3+M_4.
\end{aligned}
$$

To estimate M_3 and M_4 we have to repeat the arguments in sections 1.1, with the exception of the term $R_{20}+P_7$.

Following the same way, we will get that

$$
\begin{aligned}
M_3 \leq &\exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)}+\|z\|^2_{H^4(S)})+C\|\Im(\frac{\varpi}{A(t)})\|_{H^2(S)}\|\Lambda^{\frac{1}{2}}\partial_\alpha^4 z\|^2_{L^2(S)}\\
&-2\Re\int_{\mathbb{T}}\frac{\sigma(\gamma)}{A(t)}\partial_\alpha^4 z_j(\gamma)\cdot\Lambda(\overline{\partial_\alpha^4 z_j})(\gamma)d\alpha
\end{aligned}
$$

where $\gamma=\alpha\pm ih(t)$.

In order to avoid problems we write,

$$\sigma(\gamma)=\sigma(\alpha)+h(t)g_\pm(\alpha)$$

where $g_\pm=\frac{1}{h(t)}(\sigma(\gamma)-\sigma(\alpha))$.

Since

$$\sigma(\alpha)=\frac{\mu^2}{\kappa}BR(z,\varpi)(\alpha)\cdot\partial_\alpha^\perp z(\alpha)+g\rho^2\partial_\alpha z_1(\alpha),$$

we can write,

$$g_\pm=\pm\frac{i\mu^2}{\kappa}\int_0^1\partial_\alpha(BR(z,\varpi)\cdot\partial_\alpha^\perp z)(\gamma t+(t-1)\alpha)dt\pm ig\rho^2\int_0^1\partial_\alpha^2 z_1(\gamma t+(t-1)\alpha)dt$$

then

$$\|g_\pm\|_{H^2(S)}\leq\exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)}+\|z\|^2_{H^4(S)}).$$

Thus, we get

$$
\begin{aligned}
&-2\Re\int_{\mathbb{T}}\frac{\sigma(\gamma)}{A(t)}\partial_\alpha^4 z_j(\gamma)\Lambda(\overline{\partial_\alpha^4 z_j})(\gamma)d\alpha=-2\Re\int_{\mathbb{T}}\frac{\sigma(\alpha)}{A(t)}\partial_\alpha^4 z_j(\gamma)\Lambda(\overline{\partial_\alpha^4 z_j})(\gamma)d\alpha\\
&-2h(t)\Re\int_{\mathbb{T}}\frac{g_\pm(\alpha)}{A(t)}\partial_\alpha^4 z_j(\gamma)\Lambda(\overline{\partial_\alpha^4 z_j})(\gamma)d\alpha\equiv M_3^1+M_3^2.
\end{aligned}
$$

On the one hand, since $\Re(\frac{\sigma}{A(t)})>0$ and $2g\Lambda(g)-\Lambda(g^2)\geq 0$

$$
\begin{aligned}
M_3^1&=-2\int_{\mathbb{T}}\Re(\frac{\sigma}{A(t)})(\Re(\partial_\alpha^4 z_j)\Re(\Lambda(\partial_\alpha^4 z_j))+\Im(\partial_\alpha^4 z_j)\Im(\Lambda(\partial_\alpha^4 z_j)))d\alpha\\
&\leq\|\Lambda(\frac{\sigma}{A(t)})\|_{L^\infty(S)}\|\partial_\alpha^4 z\|^2_{L^2(S)}\leq\exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)}+\|z\|^2_{H^4(S)}).
\end{aligned}
$$

On the other hand, like in the term N_5 in section 1.1.1.1

$$
\begin{aligned}
M_3^2&=-2h(t)\Re\int_{\mathbb{T}}\Lambda^{\frac{1}{2}}(\frac{g_\pm(\alpha)}{A(t)}\partial_\alpha^4 z_j(\gamma))\Lambda^{\frac{1}{2}}(\overline{\partial_\alpha^4 z_j})(\gamma)d\alpha\\
&\leq Ch(t)\|\frac{g_\pm}{A(t)}\|_{H^2(S)}(\|\partial_\alpha^4 z\|_{L^2(S)}+\|\Lambda^{\frac{1}{2}}\partial_\alpha^4 z\|_{L^2(S)})\|\Lambda^{\frac{1}{2}}\partial_\alpha^4 z\|_{L^2(S)}
\end{aligned}
$$

$$\leq \exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)} + \|z\|^2_{H^4(S)}) + Ch(t)\exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)} + \|z\|^2_{H^4(S)})\|\Lambda^{\frac{1}{2}}\partial_\alpha^4 z\|^2_{L^2(S)}.$$

For M_1,

$$M_1 \leq \frac{h'(t)}{10}\|\Lambda^{\frac{1}{2}}\partial_\alpha^4 z\|^2_{L^2(S)}$$

and M_4,

$$\begin{aligned}
M_4 &\leq \exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)} + \|z\|^2_{H^4(S)}) \\
&+ C(\|\Im(\partial_\alpha z)\Re(\partial_\alpha z\frac{\varpi}{A^2(t)})\|_{H^2(S)} + \|\Im(\partial_\alpha z)\Im(\partial_\alpha z\frac{\varpi}{A^2(t)})\|_{H^2(S)} + \|\Im(c)\|_{H^2(S)})\|\Lambda^{\frac{1}{2}}\partial_\alpha^4 z\|^2_{L^2(S)} \\
&\leq \exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)} + \|z\|^2_{H^4(S)})\|\Lambda^{\frac{1}{2}}\partial_\alpha^4 z\|^2_{L^2(S)}.
\end{aligned}$$

Then,

$$\begin{aligned}
&\frac{d}{dt}\sum_{\pm}\int_{\mathbb{T}}|\partial_\alpha^4 z_j(\alpha \pm ih(t))|^2 d\alpha \leq \exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)} + \|z\|^2_{H^4(S)}) \\
&- 10h'(t)\int_{\mathbb{T}}\Lambda(\partial_\alpha^4 z_j)(\alpha)\overline{\partial_\alpha^4 z_j(\alpha)}d\alpha \\
&+ (\exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)} + \|z\|^2_{H^4(S)})h(t) + \frac{h'(t)}{10} + \exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)} + \|z\|^2_{H^4(S)}))\|\Lambda^{\frac{1}{2}}\partial_\alpha^4 z\|^2_{L^2(S)}.
\end{aligned}$$

Choosing,

$$h(t) = \exp(-10\int_0^t G(r)dr)[\int_0^t -10G(r)\exp(10\int_0^r G(s)ds)dr + h(0)]$$

where $G(t) = \exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)} + \|z\|^2_{H^4(S)})(t)$, we eliminate the most dangerous term. The other term in the expression above,

$$\int_{\mathbb{T}}\Lambda(\partial_\alpha^4 z_j)(\alpha)\partial_\alpha^4 z_j(\alpha)d\alpha \leq \frac{C}{h(t)}\sum_{\pm}\int_{\mathbb{T}}|\partial_\alpha^4 z_j|^2 d\alpha$$

as one sees by examining the Fourier expansion of $\partial_\alpha^4 z_j(\alpha, t)$. Thus,

$$\begin{aligned}
&|-10h'(t)\int_{\mathbb{T}}\Lambda(\partial_\alpha^4 z_j)(\alpha)\partial_\alpha^4 z_j(\alpha)d\alpha| \leq C\frac{|h'(t)|}{h(t)}(\|z\|^2_{H^4(S)} + \|\mathcal{F}(z)\|^2_{L^\infty(S)}) \\
&\leq \exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)} + \|z\|^2_{H^4(S)}).
\end{aligned}$$

And we obtain finally,

$$\frac{d}{dt}\sum_{\pm}\int_{\mathbb{T}}|\partial_\alpha^4 z(\alpha \pm ih(t))|^2 d\alpha \leq \exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)} + \|z\|^2_{H^4(S)})$$

Recovering the dependency on N in our notation we have that

$$\frac{d}{dt}\sum_{\pm}\int_{\mathbb{T}}|\partial_\alpha^4 z^{[N]}(\alpha \pm ih(t))|^2 d\alpha \leq \exp C(\|\mathcal{F}(z^{[N]})\|^2_{L^\infty(S_N)} + \|z^{[N]}\|^2_{L^2(S_N)}) \tag{1.12}$$

This estimate is true wherever $t \in [0, T_N]$, where T_N is the maximal time of existence of the solution $z^{[N]}$. In addition inequality (1.12) shows that we can extend these solutions in $H^4(S)$ up to a small enough time T independent of N and dependent on the initial data. ∎

1.5 Non-splat singularity

As we have said in the introduction, it is necessary to consider a transformed Muskat problem and we need to prove instant analyticity and decay estimates in $\tilde{\Omega}$. We will prove that the energy estimates of the Theorems 1.3.1 and 1.4.1 holds in $\tilde{\Omega}$ for solutions $\tilde{z}$ of the problem $\tilde{P}$, with $\tilde{z} \in \mathcal{C}([0,T], H^k)$ for $k \geq 4$.

1.5.1 Instant analyticity in $\tilde{\Omega}$ domain

We define

$$
\begin{aligned}
q^0 &= (0,0), \quad q^1 = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}), \quad q^2 = (-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}),\\
q^3 &= (-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}), \quad q^4 = (\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})
\end{aligned}
$$

which are the singular points of the P^{-1} conformal map. We set $z(\alpha,t)$ to hold $\tilde{z}(\alpha,t) \neq q^l$ for $l = 0,1,2,3,4$. In order to get this we fix $\overline{\Omega(0)}$ so that $\frac{dP}{dw}(w) \neq 0$ for any $w \in \overline{\Omega(0)}$ without loss of generality.

We define the energy

$$
\|\tilde{z}\|_{RT} \equiv \|\tilde{z}\|^2_{H^k(S)} + \|\mathcal{F}(\tilde{z})\|^2_{L^\infty(S)} + \frac{1}{m(Q^2\tilde{\sigma})(t) - 2\lambda - \|g\|(t)} + \sum_{l=0}^{4} \frac{1}{m(q^l)(t)}
$$

where

$$
\begin{aligned}
\|g\|(t) =& C(\|\Im(\partial_\alpha \tilde{z})\Re(\partial_\alpha \tilde{z}\frac{\tilde{\varpi}Q^2}{A^2(t)})\|_{H^2(S)} + \|\Im(\partial_\alpha \tilde{z})\Im(\partial_\alpha \tilde{z}\frac{\tilde{\varpi}Q^2}{A^2(t)})\|_{H^2(S)}\\
&+ \|\Im(\frac{\tilde{\varpi}Q^2}{A(t)})\|_{H^2(S)} + \|\Im(\tilde{c})\|_{H^2(S)})
\end{aligned}
$$

and

$$
m(Q^2\tilde{\sigma})(t) = \min_\alpha Q^2(\alpha,t)\tilde{\sigma}(\alpha,t), \quad m(q^l)(t) = \min_\alpha |\tilde{z}(\alpha,t) - q^l|.
$$

where $\tilde{\sigma}$ is as in (1.2).

Theorem 1.5.1. *Let $\tilde{z}(\alpha,t)$ be a solution of $\tilde{P}$. Then, the following estimate holds:*

$$
\frac{d}{dt}\|\tilde{z}\|_{RT} \leq \exp C(\|\tilde{z}\|_{RT})
$$

for C constant.

Remark 1.5.1. *We will show the proof for $k = 4$, being the rest of the cases analogous.*

Proof. We have to estimate

$$
\frac{d}{dt}\|\partial^4_\alpha \tilde{z}\|^2_{L^2(S)}.
$$

We quote [4] for dealing with the Q^2 term. This factor do not introduce a high order term

$$\|Q^2\|_{H^k(S)} \leq \exp C(\|\tilde{z}\|_{RT}).$$

Then we have to repeat all estimates in section 1.1, in which Q^2 is involved. We will show below how to deal with them.

We find

$$\frac{d}{dt}\|\partial_\alpha^4\tilde{z}\|^2_{L^2(S)} \leq \exp C(\|\mathcal{F}(\tilde{z})\|^2_{L^\infty(S)} + \|\tilde{z}\|^2_{H^4(S)}) + 2\lambda\|\Lambda^{\frac{1}{2}}\partial_\alpha^4\tilde{z}\|^2_{L^2(S)}$$
$$+ J_1 + J_2$$

where

$$J_1 = \Re\int_{\mathbb{T}} \overline{\partial_\alpha^4\tilde{z}(\gamma)} \cdot \partial_\alpha^4(Q^2(\gamma)BR(\tilde{z},\tilde{\varpi})(\gamma))d\alpha,$$
$$J_2 = \Re\int_{\mathbb{T}} \overline{\partial_\alpha^4\tilde{z}(\gamma)} \cdot \partial_\alpha^4(\tilde{c}(\gamma)\partial_\alpha\tilde{z}(\gamma))d\alpha.$$

We get $J_1 \leq \exp C(\|\mathcal{F}(\tilde{z})\|^2_{L^\infty(S)} + \|\tilde{z}\|^2_{H^4(S)}) + I_7$ where

$$I_7 = \Re\int_{\mathbb{T}} \overline{\partial_\alpha^4\tilde{z}(\gamma)} \cdot Q^2(\gamma)\partial_\alpha^4 BR(\tilde{z},\tilde{\varpi})(\gamma)d\alpha.$$

As in 1.1.1.1 we split $I_7 = \tilde{I}_3 + \tilde{I}_4 + \tilde{I}_5 + \tilde{I}_6 + \tilde{I}_7$ in the same way, we have

$$\tilde{I}_3 + \tilde{I}_4 + \tilde{I}_5 + \tilde{I}_6 \leq \exp C(\|\mathcal{F}(\tilde{z})\|^2_{L^\infty(S)} + \|\tilde{z}\|^2_{H^4(S)}) + C\|\Im(\frac{\tilde{\varpi}}{A(t)}Q^2)\|_{H^2(S)}\|\Lambda^{\frac{1}{2}}\partial_\alpha^4\tilde{z}\|^2_{L^2(S)}$$

and $\tilde{I}_7 \leq \exp C(\|\mathcal{F}(\tilde{z})\|^2_{L^\infty(S)} + \|\tilde{z}\|^2_{H^4(S)}) + \tilde{K}_9$ being

$$\tilde{K}_4 = \frac{1}{2}\Re\int_{\mathbb{T}} \overline{\partial_\alpha^4\tilde{z}(\gamma)} \cdot \frac{\partial_\alpha^\perp\tilde{z}(\gamma)}{|\partial_\alpha\tilde{z}|^2}H(\partial_\alpha^4\tilde{\varpi})(\gamma)Q^2(\gamma)d\alpha.$$

Identity $H(\partial_\alpha) = \Lambda$ allows us to rewrite $\tilde{K}_4$ as follows

$$\tilde{K}_4 = \frac{1}{2}\Re\int_{\mathbb{T}} \Lambda(\overline{\partial_\alpha^4\tilde{z}} \cdot \frac{\partial_\alpha^\perp\tilde{z}}{|\partial_\alpha\tilde{z}|^2}Q^2)(\gamma)\partial_\alpha^3\tilde{\varpi}(\gamma)d\alpha.$$

Using the formula of the strength of the vorticity in (1.1), we decompose $\tilde{K}_4 = \tilde{L}_{12} + \tilde{L}_{13}$

$$\tilde{L}_{12} = -\kappa g\frac{\rho^2}{\mu^2}\Re\int_{\mathbb{T}} \frac{\Lambda(\overline{\partial_\alpha^4\tilde{z}} \cdot Q^2\partial_\alpha^\perp\tilde{z})(\gamma)}{A(t)}\partial_\alpha^4(P_2^{-1}(\tilde{z}(\gamma)))d\alpha$$
$$\tilde{L}_{13} = -\frac{1}{2}\Re\int_{\mathbb{T}} \frac{\Lambda(\overline{\partial_\alpha^4\tilde{z}} \cdot Q^2\partial_\alpha^\perp\tilde{z})(\gamma)}{A(t)}\partial_\alpha^3\tilde{T}(\tilde{\varpi})(\gamma)d\alpha$$

where $\tilde{T}(\tilde{\varpi}) = -2BR(\tilde{z},\tilde{\varpi})\cdot\partial_\alpha\tilde{z}$.

The term $\tilde{L}_{13}$ can be estimate as the term L_{13} in subsection 1.1.1.2. An analogous approach provides

$$\tilde{L}_{13} \leq \exp C(\|\mathcal{F}(\tilde{z})\|^2_{L^\infty(S)} + \|\tilde{z}\|^2_{H^4(S)})$$

$$-\Re\int_{\mathbb{T}}\frac{Q^2(\gamma)BR(\tilde{z},\tilde{\varpi})(\gamma)\cdot\partial_\alpha^\perp\tilde{z}(\gamma)}{A(t)}\partial_\alpha^4\tilde{z}(\gamma)\cdot\Lambda^{\frac{1}{2}}(\overline{\partial_\alpha^4\tilde{z}})(\gamma)d\alpha. \tag{1.13}$$

For $\tilde{L}_{12}$ we consider the most singular terms: $\tilde{L}_{12}\leq\exp C(\|\mathcal{F}(\tilde{z})\|^2_{L^\infty(S)}+\|\tilde{z}\|^2_{H^4(S)})+\tilde{L}^1_{12}$ where

$$\tilde{L}^1_{12}=-\kappa g\frac{\rho^2}{\mu^2}\Re\int_{\mathbb{T}}\frac{\Lambda(\overline{\partial_\alpha^4\tilde{z}}\cdot Q^2\partial_\alpha^\perp\tilde{z})(\gamma)}{A(t)}\nabla P_2^{-1}(\tilde{z}(\gamma))\cdot\partial_\alpha^4\tilde{z}(\gamma)d\alpha.$$

Then we split $\tilde{L}^1_{12}=\tilde{M}^1_{13}+\tilde{M}^2_{13}+\tilde{M}^1_{14}+\tilde{M}^2_{14}$ by writing the component of the curve:

$$\begin{aligned}
\tilde{M}^1_{13}&=\kappa g\frac{\rho^2}{\mu^2}\Re\int_{\mathbb{T}}\frac{\Lambda(\overline{\partial_\alpha^4\tilde{z}_1}Q^2\partial_\alpha\tilde{z}_2)(\gamma)}{A(t)}\partial_{\tilde{x}_1}P_2^{-1}(\tilde{z}(\gamma))\partial_\alpha^4\tilde{z}_1(\gamma)d\alpha,\\
\tilde{M}^2_{13}&=\kappa g\frac{\rho^2}{\mu^2}\Re\int_{\mathbb{T}}\frac{\Lambda(\overline{\partial_\alpha^4\tilde{z}_1}Q^2\partial_\alpha\tilde{z}_2)(\gamma)}{A(t)}\partial_{\tilde{x}_2}P_2^{-1}(\tilde{z}(\gamma))\partial_\alpha^4\tilde{z}_2(\gamma)d\alpha,\\
\tilde{M}^1_{14}&=-\kappa g\frac{\rho^2}{\mu^2}\Re\int_{\mathbb{T}}\frac{\Lambda(\overline{\partial_\alpha^4\tilde{z}_2}Q^2\partial_\alpha\tilde{z}_1)(\gamma)}{A(t)}\partial_{\tilde{x}_1}P_2^{-1}(\tilde{z}(\gamma))\partial_\alpha^4\tilde{z}_1(\gamma)d\alpha,\\
\tilde{M}^2_{14}&=-\kappa g\frac{\rho^2}{\mu^2}\Re\int_{\mathbb{T}}\frac{\Lambda(\overline{\partial_\alpha^4\tilde{z}_2}Q^2\partial_\alpha\tilde{z}_1)(\gamma)}{A(t)}\partial_{\tilde{x}_2}P_2^{-1}(\tilde{z}(\gamma))\partial_\alpha^4\tilde{z}_2(\gamma)d\alpha.
\end{aligned}$$

The commutator estimate yields

$$\begin{aligned}
&\tilde{M}^1_{13}\leq\exp C(\|\mathcal{F}(\tilde{z})\|^2_{L^\infty(S)}+\|\tilde{z}\|^2_{H^4(S)})\\
&+\kappa g\frac{\rho^2}{\mu^2}\Re\int_{\mathbb{T}}\frac{Q^2(\gamma)\partial_{\tilde{x}_1}P_2^{-1}(\tilde{z}(\gamma))}{A(t)}\partial_\alpha\tilde{z}_2(\gamma)\partial_\alpha^4\tilde{z}_1(\gamma)\Lambda(\overline{\partial_\alpha^4\tilde{z}_1})(\gamma)d\alpha, &(1.14)\\
&\tilde{M}^2_{13}\leq\exp C(\|\mathcal{F}(\tilde{z})\|^2_{L^\infty(S)}+\|\tilde{z}\|^2_{H^4(S)})\\
&+\kappa g\frac{\rho^2}{\mu^2}\Re\int_{\mathbb{T}}\frac{Q^2(\gamma)\partial_{\tilde{x}_2}P_2^{-1}(\tilde{z}(\gamma))}{A(t)}\partial_\alpha\tilde{z}_2(\gamma)\partial_\alpha^4\tilde{z}_2(\gamma)\Lambda(\overline{\partial_\alpha^4\tilde{z}_1})(\gamma)d\alpha,\\
&\tilde{M}^1_{14}\leq\exp C(\|\mathcal{F}(\tilde{z})\|^2_{L^\infty(S)}+\|\tilde{z}\|^2_{H^4(S)})\\
&-\kappa g\frac{\rho^2}{\mu^2}\Re\int_{\mathbb{T}}\frac{Q^2(\gamma)\partial_{\tilde{x}_1}P_2^{-1}(\tilde{z}(\gamma))}{A(t)}\partial_\alpha\tilde{z}_1(\gamma)\partial_\alpha^4\tilde{z}_1(\gamma)\Lambda(\overline{\partial_\alpha^4\tilde{z}_2})(\gamma)d\alpha,\\
&\tilde{M}^2_{14}\leq\exp C(\|\mathcal{F}(\tilde{z})\|^2_{L^\infty(S)}+\|\tilde{z}\|^2_{H^4(S)})\\
&-\kappa g\frac{\rho^2}{\mu^2}\Re\int_{\mathbb{T}}\frac{Q^2(\gamma)\partial_{\tilde{x}_2}P_2^{-1}(\tilde{z}(\gamma))}{A(t)}\partial_\alpha\tilde{z}_1(\gamma)\partial_\alpha^4\tilde{z}_2(\gamma)\Lambda(\overline{\partial_\alpha^4\tilde{z}_2})(\gamma)d\alpha. &(1.15)
\end{aligned}$$

Using that

$$\partial_\alpha\tilde{z}_2\partial_\alpha^4\tilde{z}_2=-3\partial_\alpha^2\tilde{z}\cdot\partial_\alpha^3\tilde{z}-\partial_\alpha\tilde{z}_1\partial_\alpha^4\tilde{z}_1,$$

we get

$$\begin{aligned}
&\tilde{M}^2_{13}\leq\exp C(\|\mathcal{F}(\tilde{z})\|^2_{L^\infty(S)}+\|\tilde{z}\|^2_{H^4(S)})\\
&-\kappa g\frac{\rho^2}{\mu^2}\Re\int_{\mathbb{T}}\frac{Q^2(\gamma)\partial_{\tilde{x}_2}P_2^{-1}(\tilde{z}(\gamma))}{A(t)}\partial_\alpha\tilde{z}_1(\gamma)\partial_\alpha^4\tilde{z}_1(\gamma)\Lambda(\overline{\partial_\alpha^4\tilde{z}_1})(\gamma)d\alpha, &(1.16)\\
&\tilde{M}^1_{14}\leq\exp C(\|\mathcal{F}(\tilde{z})\|^2_{L^\infty(S)}+\|\tilde{z}\|^2_{H^4(S)})\\
&+\kappa g\frac{\rho^2}{\mu^2}\Re\int_{\mathbb{T}}\frac{Q^2(\gamma)\partial_{\tilde{x}_1}P_2^{-1}(\tilde{z}(\gamma))}{A(t)}\partial_\alpha\tilde{z}_2(\gamma)\partial_\alpha^4\tilde{z}_2(\gamma)\Lambda(\overline{\partial_\alpha^4\tilde{z}_2})(\gamma)d\alpha. &(1.17)
\end{aligned}$$

Adding the inequalities (1.14),(1.16),(1.17) and (1.15) it is easy to check

$$\begin{aligned}\tilde{L}_{12} \leq & \exp C(\|\mathcal{F}(\tilde{z})\|^2_{L^\infty(S)} + \|\tilde{z}\|^2_{H^4(S)})\\ & -\kappa g\frac{\rho^2}{\mu^2}\Re\int_{\mathbb{T}}\frac{Q^2(\gamma)\nabla P_2^{-1}(\tilde{z}(\gamma))}{A(t)}\cdot\partial_\alpha^\perp\tilde{z}(\gamma)\partial_\alpha^4\tilde{z}(\gamma)\cdot\Lambda(\partial_\alpha^4\tilde{z})(\gamma)d\alpha.\end{aligned}$$

Above inequality together with (1.13) let us obtain

$$\begin{aligned}\tilde{K}_4 \leq & \exp C(\|\mathcal{F}(\tilde{z})\|^2_{L^\infty(S)} + \|\tilde{z}\|^2_{H^4(S)})\\ & -\Re\int_{\mathbb{T}}\frac{Q^2(\gamma)\tilde{\sigma}(\gamma)}{A(t)}\partial_\alpha^4\tilde{z}(\gamma)\cdot\Lambda(\partial_\alpha^4\tilde{z})(\gamma)d\alpha\end{aligned}$$

with $\tilde{\sigma}$ given in (1.2).

Considering $m(Q^2\tilde{\sigma})(t)$ and the pointwise inequality $2f\Lambda(f)\geq\Lambda(f^2)$ we check

$$\tilde{I}_7 \leq \exp C(\|\mathcal{F}(\tilde{z})\|^2_{L^\infty(S)} + \|\tilde{z}\|^2_{H^4(S)}) - m(Q^2\tilde{\sigma})(t)\|\partial_\alpha^4\tilde{z}\|^2_{L^2(S)}.$$

For J_2 it is easy to deal with $\partial_\alpha^4\tilde{c}$ in the same way as in section 1.1.1.3. The analogous approach provides

$$\begin{aligned}J_2 \leq & \exp C(\|\mathcal{F}(\tilde{z})\|^2_{L^\infty(S)} + \|\tilde{z}\|^2_{H^4(S)})\\ & + C(\|\Im(\partial_\alpha\tilde{z})\Re(\partial_\alpha\tilde{z}\frac{\tilde{\varpi}Q^2}{A^2(t)})\|_{H^2(S)} + \|\Im(\partial_\alpha\tilde{z})\Im(\partial_\alpha\tilde{z}\frac{\tilde{\varpi}Q^2}{A^2(t)})\|_{H^2(S)}\\ & + \|\Im(\tilde{c})\|_{H^2(S)})\|\Lambda^{\frac{1}{2}}\partial_\alpha^4\tilde{z}\|^2_{L^2(S)}.\end{aligned}$$

Finally we obtain,

$$\begin{aligned}\frac{d}{dt}\|\partial_\alpha^4\tilde{z}\|^2_{L^2(S)} \leq & \exp C(\|\mathcal{F}(\tilde{z})\|^2_{L^\infty(S)} + \|\tilde{z}\|^2_{H^4(S)})\\ & + (2\lambda + \|g\| - m(Q^2\tilde{\sigma}))\|\Lambda^{\frac{1}{2}}\partial_\alpha^4\tilde{z}\|^2_{L^2(S)}.\end{aligned}$$

Bearing in mind the singular points of the P^{-1} together with the estimation for $m(Q^2\tilde{\sigma})(t)$, which we can obtain in analogous way as in section 1.2, we have the desired estimate. ■

1.5.2 Decay of the strip of analyticity in the $\tilde{\Omega}$ domain

Theorem 1.5.2. *Let $\tilde{z}(\alpha,0)=\tilde{z}^0(\alpha)$ be an analytic curve in the strip*

$$S=\{\alpha+i\varsigma\in\mathbb{C}:|\varsigma|<h(0)\},$$

with $h(0)>0$ and satisfying:

* *The arc-chord condition, $\mathcal{F}(\tilde{z}^0)(\alpha+i\varsigma,\beta)\in L^\infty(S\times\mathbb{R})$*
* *The curve $\tilde{z}^0(\alpha)$ is real for real α*
* *The functions $\tilde{z}_1^0(\alpha)-\alpha$ and $\tilde{z}_2{}^0(\alpha)$ are periodic with period 2π*
* *The functions $\tilde{z}_1{}^0(\alpha)-\alpha$ and $\tilde{z}_2{}^0(\alpha)$ belong to $H^4(S)$*

Then there exist a time T and a solution of the Muskat problem in $\tilde{\Omega}$, $\tilde{z}(\alpha,t)$ defined for $0<t\leq T$ that continues analytically into some complex strip for each fixed $t\in[0,T]$. Here T is either a small constant depending only on $\exp C(\|\mathcal{F}(\tilde{z}^0)\|^2_{L^\infty(S)}+\|\tilde{z}^0\|^2_{L^2(S)})$.

Proof. Here we proceed in the same way that in the proof of the Theorem 1.4.1.

After we use the Galerkin approximation, by Lemma 1.4.1 we get

$$\frac{d}{dt}\sum_{\pm}\int_{\mathbb{T}}|\partial^4_\alpha\tilde{z}_j(\alpha\pm ih(t))|^2d\alpha\leq\frac{h'(t)}{10}\sum_{\pm}\int_{\mathbb{T}}\Lambda(\partial^4_\alpha\tilde{z}_j)(\alpha\pm ih(t))\overline{\partial^4_\alpha\tilde{z}_j(\alpha\pm ih(t))}$$
$$-10h'(t)\int_{\mathbb{T}}\Lambda(\partial^4_\alpha\tilde{z}_j)(\alpha)\overline{\partial^4_\alpha\tilde{z}_j(\alpha)}d\alpha+2\sum_{\pm}\Re\int_{\mathbb{T}}\partial^4_\alpha(Q^2BR(\tilde{z}_j,\tilde{\varpi})(\alpha\pm ih(t))\overline{\partial^4_\alpha z_j(\alpha\pm ih(t))}$$
$$+2\sum_{\pm}\Re\int_{\mathbb{T}}\partial^4_\alpha(\tilde{c}(\alpha\pm ih(t))\partial_\alpha\tilde{z}_j(\alpha\pm ih(t)))\overline{\partial^4_\alpha\tilde{z}_j(\alpha\pm ih(t))}.$$

We write,

$$Q^2(\gamma)\tilde{\sigma}(\gamma)=Q^2(\alpha)\tilde{\sigma}(\alpha)+h(t)\tilde{g_\pm}(\alpha)$$

and we have

$$\frac{d}{dt}\sum_{\pm}\int_{\mathbb{T}}|\partial^4_\alpha\tilde{z}_j(\alpha\pm ih(t))|^2d\alpha\leq\exp C(\|\mathcal{F}(\tilde{z})\|^2_{L^\infty(S)}+\|\tilde{z}\|^2_{H^4(S)})$$
$$-10h'(t)\int_{\mathbb{T}}\Lambda(\partial^4_\alpha\tilde{z}_j)(\alpha)\overline{\partial^4_\alpha\tilde{z}_j(\alpha)}d\alpha$$
$$+(\exp C(\|\mathcal{F}(\tilde{z})\|^2_{L^\infty(S)}+\|\tilde{z}\|^2_{H^4(S)})h(t)+\frac{h'(t)}{10}+\exp C(\|\mathcal{F}(\tilde{z})\|^2_{L^\infty(S)}+\|\tilde{z}\|^2_{H^4(S)}))\|\Lambda^{\frac{1}{2}}\partial^4_\alpha\tilde{z}\|^2_{L^2(S)}.$$

Choosing,

$$h(t)=\exp(-10\int_0^tG(r)dr)[\int_0^t-10G(r)\exp(10\int_0^rG(s)ds)dr+h(0)]\tag{1.18}$$

where $G(t)=\exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)}+\|z\|^2_{H^4(S)})(t)$ we get the desired estimation. ■

1.5.3 Proof of Theorem 0.0.5

Let $z_0(\alpha)\in H^4$, from Theorem 1.3 there exists a local solution z that becomes real-analytic in the complex strip S(t).

Suppose that there exists a time T where we have a splat singularity, i.e., the smooth interface collapses along an arc at time T.

From Theorem 1.4.1, our strip of analyticity is nonzero as long as the regularity of the curve and the arc-chord condition do not fail. But at splat time T, the arc-chord condition blows-up, and we cannot guarantee analyticity at that time.

At this point, we transform the system to the tilde domain $\tilde{\Omega}$.

As long as the regularity of the curve and the arc-chord condition do not fail, from Theorem 1.5.1 we have

$$\frac{d}{dt}\|\tilde{z}\|_{RT}\leq\exp C(\|\tilde{z}\|_{RT})$$

where the constant C only depends on the initial data and

$$\|\tilde{z}\|_{RT} \equiv \|\tilde{z}\|^2_{H^k(S)} + \|\mathcal{F}(\tilde{z})\|^2_{L^\infty(S)} + \frac{1}{m(Q^2\tilde{\sigma})(t) - 2\lambda - \|g\|(t)} + \sum_{l=0}^{4} \frac{1}{m(q^l)(t)}.$$

Hence, we can conclude that our transformed curve $\tilde{z}$ is real-analytic into the strip $S(t)$. From the proof of Theorem 1.5.2, this complex strip decays exponentially until a time that depends on the regularity of the curve and the arc-chord condition too [see equation (1.18)].

Since in $\tilde{\Omega}$ the arc-chord condition and the regularity of the curve are bounded, the strip of analyticity is nonzero and therefore we can guarantee the analyticity at time T.

Thus, applying P^{-1}, we have that the analytic curve self-intersects along an arc, therefore we get a contradiction and hence Theorem 0.0.5 is proved.

Chapter 2

Local-existence for the Inhomogeneous Muskat problem

In this chapter we study the evolution of the interface between two different fluids in a porous media with two different permeabilities. We prove local existence in Sobolev spaces, when the free boundary is given by the discontinuity among the densities and viscosities of the fluids. Recall that the equations for this problem are:

$$IP = \begin{cases} z_t(\alpha,t) = BR(\varpi_1, z)_z(\alpha,t) + BR(\varpi_2, h)_z(\alpha,t) + c(\alpha,t)\partial_\alpha z(\alpha,t) \\ c(\alpha,t) = \frac{\alpha+\pi}{2\pi A(t)} \int_{\mathbb{T}} \partial_\alpha z(\beta,t) \cdot \partial_\alpha (BR(\varpi_1,z)_z + BR(\varpi_2,h)_z) d\beta \\ - \int_{-\pi}^{\alpha} \frac{\partial_\alpha z(\beta,t)}{A(t)} \cdot \partial_\alpha (BR(\varpi_1,z)_z + BR(\varpi_2,h)_z) d\beta \\ \varpi_1(\alpha,t) = -2\frac{\mu^2-\mu^1}{\mu^2+\mu^1}(BR(\varpi_1,z)_z + BR(\varpi_2,h)_z) \cdot \partial_\alpha z(\alpha,t) - 2\kappa^1 \frac{\rho^2-\rho^1}{\mu^2+\mu^1} g \partial_\alpha z_2(\alpha,t) \\ \varpi_2(\alpha,t) = -2\frac{\kappa^1-\kappa^2}{\kappa^2+\kappa^1}(BR(\varpi_1,z)_h + BR(\varpi_2,h)_h) \cdot \partial_\alpha h(\alpha) \end{cases}$$

where the Rayleigh-Taylor condition can be written as follows:

$$\sigma(\alpha,t) = \frac{\mu^2-\mu^1}{\kappa^1}(BR(\varpi_1,z)_z + BR(\varpi_2,h)_z) \cdot \partial_\alpha^\perp z(\alpha) + (\rho^2-\rho^1) g \partial_\alpha z_1(\alpha) > 0.$$

2.1 Inverse Operator

In this section we are going to study the operator which will allow us to estimate the H^k-norm of the strengths of the vorticity. In order to do that we need begin estimating the L^2-norm of the inverse operator and then get the estimations of the $H^{\frac{1}{2}}$-norm.

2.1.1 The basic operator

First, let us present the operator we work with and prove some results needed before.

$$\mathcal{T}(u_1,u_2)(\alpha) = \begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \tag{2.1}$$

where

$$T_1(u)(\alpha) = 2BR(u,z)_z(\alpha) \cdot \partial_\alpha z(\alpha)$$
$$T_2(u)(\alpha) = 2BR(u,h)_z(\alpha) \cdot \partial_\alpha z(\alpha)$$

$$T_3(u)(\alpha) = 2BR(u,z)_h(\alpha) \cdot \partial_\alpha h(\alpha)$$
$$T_4(u)(\alpha) = 2BR(u,h)_h(\alpha) \cdot \partial_\alpha h(\alpha)$$

Lemma 2.1.1. *Suppose that* $\|\mathcal{F}(z)\|_{L^\infty} < \infty$*,* $\|\mathcal{F}(h)\|_{L^\infty} < \infty$*,* $\|d(z,h)\|_{L^\infty} < \infty$ *and* $z \in \mathcal{C}^{2,\delta}$*,* $h \in \mathcal{C}^{2,\delta}$*, where*

$$d(z,h) = \frac{1}{|z(\alpha) - h(\alpha-\beta)|^2}.$$

Then $\mathcal{T} : L^2 \times L^2 \to H^1 \times H^1$ *is compact and*

$$\|\mathcal{T}\|_{L^2\times L^2\to H^1\times H^1} \leq C\|\mathcal{F}(z)\|^2_{L^\infty}\|d(z,h)\|^2_{L^\infty}\|z\|^4_{\mathcal{C}^{2,\delta}}.$$

Proof. We have

$$\mathcal{T}(w)(\alpha) = \begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} T_1(u) + T_2(v) \\ T_3(u) + T_4(v) \end{pmatrix}$$

and we consider $\|(u,v)\|_{L^2} = \|u\|_{L^2} + \|v\|_{L^2}$, then

$$\|\mathcal{T}(w)\|_{L^2} = \|T_1(u) + T_2(v)\|_{L^2} + \|T_3(u) + T_4(v)\|_{L^2}$$

We want to estimate $\|\partial_\alpha \mathcal{T}(w)\|_{L^2}$. Since

$$\|\partial_\alpha T_1(u) + \partial_\alpha T_2(v)\|_{L^2} \leq \|\partial_\alpha T_1(u)\|_{L^2} + \|\partial_\alpha T_2(v)\|_{L^2},$$
$$\|\partial_\alpha T_3(u) + \partial_\alpha T_4(v)\|_{L^2} \leq \|\partial_\alpha T_3(u)\|_{L^2} + \|\partial_\alpha T_4(v)\|_{L^2},$$

it is enough to estimate each T_i for $i = 1,2,3,4$ separately.

Operator T_1 and T_4 are exactly the same as the operator T on [11]. Therefore, by lemma 3.1 on [11] we have:

$$\|\partial_\alpha T_1(u)\|_{L^2} \leq C\|\mathcal{F}(z)\|^2_{L^\infty}\|z\|^4_{\mathcal{C}^{2,\delta}}\|u\|_{L^2},$$
$$\|\partial_\alpha T_4(v)\|_{L^2} \leq C\|\mathcal{F}(h)\|^2_{L^\infty}\|h\|^4_{\mathcal{C}^{2,\delta}}\|v\|_{L^2}.$$

Let us estimate operator T_2 and T_3. We write first,

$$\begin{aligned} \partial_\alpha T_2(v) = &\frac{1}{\pi}PV\int_{\mathbb{R}} \partial_\alpha\Big(\frac{(z(\alpha) - h(\alpha-\beta))^\perp \cdot \partial_\alpha z(\alpha)}{|z(\alpha) - h(\alpha-\beta)|^2}\Big)v(\alpha-\beta)d\beta \\ &+ \frac{1}{\pi}PV\int_{\mathbb{R}} \frac{(z(\alpha) - h(\alpha-\beta))^\perp \cdot \partial_\alpha z(\alpha)}{|z(\alpha) - h(\alpha-\beta)|^2}\partial_\alpha v(\alpha-\beta)d\beta \equiv I_1 + I_2. \end{aligned}$$

Then, using $\Delta zh = z(\alpha) - h(\alpha-\beta)$ in order to reduce notation,

$$\begin{aligned} I_1 = &\frac{1}{\pi}PV\int_{\mathbb{R}} \frac{(\partial_\alpha z(\alpha) - \partial_\alpha h(\alpha-\beta))^\perp \cdot \partial_\alpha z(\alpha)}{|z(\alpha) - h(\alpha-\beta)|^2}v(\alpha-\beta)d\beta \\ &+ \frac{1}{\pi}PV\int_{\mathbb{R}} \frac{(z(\alpha) - h(\alpha-\beta))^\perp \cdot \partial^2_\alpha z(\alpha)}{|z(\alpha) - h(\alpha-\beta)|^2}v(\alpha-\beta)d\beta \\ &- \frac{2}{\pi}PV\int_{\mathbb{R}} \frac{(\Delta zh)^\perp \cdot \partial_\alpha z(\alpha)(\Delta zh)\cdot\partial_\alpha \Delta zh}{|z(\alpha) - h(\alpha-\beta)|^4}v(\alpha-\beta)d\beta \\ &\equiv I_1^1 + I_1^2 + I_1^3. \end{aligned}$$

Since $\partial_\alpha z(\alpha)\cdot\partial_\alpha z(\alpha)^\perp = 0$ we have,

$$I_1^1 = -\frac{1}{\pi}PV\int_{\mathbb{R}}\frac{\partial_\alpha h(\alpha-\beta)\cdot\partial_\alpha z(\alpha)}{|z(\alpha)-h(\alpha-\beta)|^2}v(\alpha-\beta)d\beta \leq C\|d(z,h)\|_{L^\infty}\|\partial_\alpha z\|_{L^\infty}\|h\|_{H^1}\|v\|_{L^2}.$$

Using the Cauchy inequality it is easy to get $u\cdot v\leq \frac{|u|^2}{2}+\frac{|v|^2}{2}$, then

$$\begin{aligned}I_1^2 &\leq \frac{1}{2\pi}PV\int_{\mathbb{R}}v(\alpha-\beta)d\beta + \frac{1}{2\pi}PV\int_{\mathbb{R}}\frac{|\partial_\alpha^2 z(\alpha)|^2}{|z(\alpha)-h(\alpha-\beta)|^2}v(\alpha-\beta)d\beta\\ &\leq C\|v\|_{L^2}+C\|d(z,h)\|_{L^\infty}\|z\|^2_{\mathcal{C}^2}\|v\|_{L^2}\end{aligned}$$

$$\begin{aligned}|I_1^3| &\leq \frac{2}{\pi}PV\int_{\mathbb{R}}\frac{|z(\alpha)-h(\alpha-\beta)|^2|\partial_\alpha z(\alpha)||\partial_\alpha z(\alpha)-\partial_\alpha h(\alpha-\beta)|}{|z(\alpha)-h(\alpha-\beta)|^4}|v(\alpha-\beta)|d\beta\\ &\leq C\|d(z,h)\|_{L^\infty}\|z\|_{\mathcal{C}^1}(\|z\|_{\mathcal{C}^1}+\|h\|_{\mathcal{C}^1})\|v\|_{L^2}\end{aligned}$$

On the other hand, using integration by parts

$$\begin{aligned}I_2 &= \frac{1}{\pi}PV\int_{\mathbb{R}}\partial_\beta\Big(\frac{(z(\alpha)-h(\alpha-\beta))^\perp\cdot\partial_\alpha z(\alpha)}{|z(\alpha)-h(\alpha-\beta)|^2}\Big)v(\alpha-\beta)d\beta\\ &= \frac{1}{\pi}PV\int_{\mathbb{R}}\frac{(\partial_\alpha h(\alpha-\beta))^\perp\cdot\partial_\alpha z(\alpha)}{|z(\alpha)-h(\alpha-\beta)|^2}v(\alpha-\beta)d\beta\\ &-\frac{2}{\pi}PV\int_{\mathbb{R}}\frac{(z(\alpha)-h(\alpha-\beta))^\perp\cdot\partial_\alpha z(\alpha)(z(\alpha)-h(\alpha-\beta))\cdot\partial_\alpha h(\alpha-\beta)}{|z(\alpha)-h(\alpha-\beta)|^4}v(\alpha-\beta)d\beta\\ &\leq C\|d(z,h)\|_{L^\infty}\|\partial_\alpha z\|_{L^\infty}\|h\|_{\mathcal{C}^1}\|v\|_{L^2}+C\|d(z,h)\|_{L^\infty}\|z\|_{\mathcal{C}^1}\|h\|_{\mathcal{C}^1}\|v\|_{L^2}.\end{aligned}$$

Then,

$$\|\partial_\alpha T_2(v)\|_{L^2}\leq C\|d(z,h)\|_{L^\infty}\|z\|^2_{\mathcal{C}^2}\|h\|_{\mathcal{C}^1}\|v\|_{L^2}.$$

Finally, we have to estimate $\partial_\alpha T_3$. We have,

$$\begin{aligned}\partial_\alpha T_3(u)(\alpha) &= \frac{1}{\pi}PV\int_{\mathbb{R}}\partial_\alpha\Big(\frac{(h(\alpha)-z(\alpha-\beta))^\perp\cdot\partial_\alpha h(\alpha)}{|h(\alpha)-z(\alpha-\beta)|^2}\Big)u(\alpha-\beta)d\beta\\ &+\frac{1}{\pi}PV\int_{\mathbb{R}}\frac{(h(\alpha)-z(\alpha-\beta))^\perp\cdot\partial_\alpha h(\alpha)}{|h(\alpha)-z(\alpha-\beta)|^2}\partial_\alpha u(\alpha-\beta)d\beta.\end{aligned}$$

Changing z for h, we can check that we have the same estimates as in T_2. Thus,

$$\|T_3(u)\|_{L^2}\leq C\|d(z,h)\|_{L^\infty}\|h\|^2_{\mathcal{C}^2}\|z\|_{\mathcal{C}^1}\|u\|_{L^2}.$$

Therefore,

$$\|\partial_\alpha\mathcal{T}(u,v)\|_{L^2}\leq C\|\mathcal{F}(z)\|^2_{L^\infty}\|\mathcal{F}(h)\|^2_{L^\infty}\|d(z,h)\|^2_{L^\infty}\|h\|^4_{\mathcal{C}^{2,\delta}}\|z\|^4_{\mathcal{C}^{2,\delta}}\|(u,v)\|_{L^2}.$$

Since h is fixed on time, $\|\mathcal{F}(h)\|^2_{L^\infty}$ and $\|h\|^4_{\mathcal{C}^{2,\delta}}$ are not dependent of time. Thus we get,

$$\|\partial_\alpha\mathcal{T}(w)\|_{L^2}\leq C\|\mathcal{F}(z)\|^2_{L^\infty}\|d(z,h)\|^2_{L^\infty}\|z\|^4_{\mathcal{C}^{2,\delta}}\|(w)\|_{L^2}.$$

■

2.1.2 Estimates on the inverse operator

We are going to work with the adjoint operator of $\mathcal{T}$ in order to estimate the inverse operator $(I+M\mathcal{T})^{-1}$. We have,

$$\begin{aligned}&\left(\begin{pmatrix}u_1\\u_2\end{pmatrix},\begin{pmatrix}T_1 & T_2\\T_3 & T_4\end{pmatrix}\begin{pmatrix}w_1\\w_2\end{pmatrix}\right)=\left(\begin{pmatrix}u_1\\u_2\end{pmatrix},\begin{pmatrix}T_1(w_1)+T_2(w_2)\\T_3(w_1)+T_4(w_2)\end{pmatrix}\right)\\&=(T_1(w_1),u_1)+(T_2(w_2),u_1)+(T_3(w_1),u_2)+(T_4(w_2),u_2)\\&=(w_1,T_1^*(u_1))+(w_2,T_2^*(u_1))+(w_1,T_3^*(u_2))+(w_2,T_4^*(u_2))=\\&\left(\begin{pmatrix}w_1\\w_2\end{pmatrix},\begin{pmatrix}T_1^*(u_1)+T_3^*(u_2)\\T_2^*(u_1)+T_4^*(u_2)\end{pmatrix}\right)=\left(\begin{pmatrix}w_1\\w_2\end{pmatrix},\begin{pmatrix}T_1^* & T_3^*\\T_2^* & T_4^*\end{pmatrix}\begin{pmatrix}u_1\\u_2\end{pmatrix}\right)\end{aligned}$$

The adjoint operator is given by

$$\mathcal{T}^*(u_1,u_2)(\alpha)=\begin{pmatrix}T_1^* & T_3^*\\T_2^* & T_4^*\end{pmatrix}\begin{pmatrix}u_1\\u_2\end{pmatrix}$$

where we can compute:

$$T_1^*(u)(\alpha)=-\frac{1}{\pi}PV\int_{\mathbb{R}}\frac{(z(\alpha)-z(\beta))^{\perp}\cdot\partial_\alpha z(\beta)}{|z(\alpha)-z(\beta)|^2}u(\beta)d\beta,$$

$$T_2^*(u)(\alpha)=-\frac{1}{\pi}PV\int_{\mathbb{R}}\frac{(h(\alpha)-z(\beta))^{\perp}\cdot\partial_\alpha z(\beta)}{|h(\alpha)-z(\beta)|^2}u(\beta)d\beta,$$

$$T_3^*(u)(\alpha)=-\frac{1}{\pi}PV\int_{\mathbb{R}}\frac{(z(\alpha)-h(\beta))^{\perp}\cdot\partial_\alpha h(\beta)}{|z(\alpha)-h(\beta)|^2}u(\beta)d\beta,$$

and

$$T_4^*(u)(\alpha)=-\frac{1}{\pi}PV\int_{\mathbb{R}}\frac{(h(\alpha)-h(\beta))^{\perp}\cdot\partial_\alpha h(\beta)}{|h(\alpha)-h(\beta)|^2}u(\beta)d\beta.$$

Proposition 2.1.1. *Suppose that* $\|\mathcal{F}(z)\|_{L^\infty}<\infty$, $\|\mathcal{F}(h)\|_{L^\infty}<\infty$, $\|d(z,h)\|_{L^\infty}<\infty$ *and* $z,h\in\mathcal{C}^{2,\delta}$. *Then* $\mathcal{T}^*:L^2\times L^2\to H^1\times H^1$ *and*

$$\|\mathcal{T}^*\|_{L^2\times L^2\to H^1\times H^1}\leq\|\mathcal{F}(z)\|^2_{L^\infty}\|d(z,h)\|^2_{L^\infty}\|z\|^2_{\mathcal{C}^{2,\delta}}.$$

Proof. In the same way as in the study of $\mathcal{T}$, we can prove this estimate studying each T_i^*.

$$T_1^*(u)=-\frac{1}{\pi}PV\int_{\mathbb{R}}\frac{(z(\alpha)-z(\beta))^{\perp}\cdot\partial_\alpha z(\beta)}{|z(\alpha)-z(\beta)|^2}u(\beta)d\beta$$

then

$$\begin{aligned}\partial_\alpha T_1^*(u)=&-\frac{1}{\pi}PV\int_{\mathbb{R}}\partial_\alpha\Big(\frac{(\Delta z)^{\perp}\cdot\partial_\alpha z(\alpha-\beta)}{|\Delta z|^2}\Big)u(\alpha-\beta)d\beta\\&-\frac{1}{\pi}PV\int_{\mathbb{R}}\frac{(\Delta z)^{\perp}\cdot\partial_\alpha z(\alpha-\beta)}{|\Delta z|^2}\partial_\alpha u(\alpha-\beta)d\beta\equiv I_1+I_2.\end{aligned}$$

I_1 is estimated in the same way that operator T_1. Using integration by parts

$$I_2=\frac{1}{\pi}PV\int_{\mathbb{R}}\frac{(\Delta z)^{\perp}\cdot\partial_\alpha z(\alpha-\beta)}{|\Delta z|^2}\partial_\beta u(\alpha-\beta)d\beta$$

$$
\begin{aligned}
&= -\frac{1}{\pi} PV \int_{\mathbb{R}} \partial_\beta \Big(\frac{(\Delta z)^\perp \cdot \partial_\alpha z(\alpha-\beta)}{|\Delta z|^2}\Big) u(\alpha-\beta) d\beta \\
&= -\frac{1}{\pi} PV \int_{\mathbb{R}} \frac{(\partial_\alpha^\perp z(\alpha-\beta) \cdot \partial_\alpha z(\alpha-\beta)}{|\Delta z|^2} u(\alpha-\beta) d\beta \\
&- \frac{1}{\pi} PV \int_{\mathbb{R}} \frac{(\Delta z)^\perp \cdot \partial_\alpha^2 z(\alpha-\beta)}{|\Delta z|^2} u(\alpha-\beta) d\beta \\
&+ \frac{2}{\pi} PV \int_{\mathbb{R}} \frac{(\Delta z)^\perp \cdot \partial_\alpha z(\alpha-\beta) \Delta z \cdot \partial_\alpha z(\alpha-\beta)}{|\Delta z|^2}) u(\alpha-\beta) d\beta \equiv I_2^1 + I_2^2 + I_2^3.
\end{aligned}
$$

Since $\partial_\alpha^\perp z \cdot \partial_\alpha z = 0$, $I_2^1 = 0$. We can write

$$
\begin{aligned}
I_2^2 = &-\frac{1}{\pi} PV \int_{\mathbb{R}} \Big(\frac{(\Delta z)^\perp}{|\Delta z|^2} - \frac{\partial_\alpha^\perp z(\alpha)}{\beta |\partial_\alpha z(\alpha)|^2}\Big) \cdot \partial_\alpha^2 z(\alpha-\beta) u(\alpha-\beta) d\beta \\
&- \frac{1}{\pi} PV \int_{\mathbb{R}} \frac{\partial_\alpha^\perp z(\alpha)}{\beta |\partial_\alpha z(\alpha)|^2} \cdot \partial_\alpha^2 z(\alpha-\beta) u(\alpha-\beta) d\beta \equiv I_2^{21} + I_2^{22}.
\end{aligned}
$$

Since we compute:

$$
\begin{aligned}
&\frac{(\Delta z)^\perp}{|\Delta z|^2} - \frac{\partial_\alpha^\perp z(\alpha)}{\beta |\partial_\alpha z(\alpha)|^2} = \frac{\beta |\partial_\alpha z(\alpha)|^2 \Delta z^\perp - \partial_\alpha^\perp z(\alpha) |\Delta z|^2}{\beta |\partial_\alpha z(\alpha)|^2 |\Delta z|^2} \\
&= \frac{\beta^2 |\partial_\alpha z(\alpha)|^2 \int_0^1 \partial_\alpha^\perp z(\alpha-\beta+t\beta) dt - \partial_\alpha^\perp z(\alpha) |\Delta z|^2}{\beta |\partial_\alpha z(\alpha)|^2 |\Delta z|^2} \\
&= \frac{\beta^3 |\partial_\alpha z(\alpha)|^2 \int_0^1 \int_0^1 \partial_\alpha^{2\perp} z(\alpha - s\beta + st\beta)(t-1) ds dt + \partial_\alpha^\perp z(\alpha)(\beta^2 |\partial_\alpha z(\alpha)|^2 - |\Delta z|^2)}{\beta |\partial_\alpha z(\alpha)|^2 |\Delta z|^2} \\
&\frac{\beta^2 \int_0^1 \int_0^1 \partial_\alpha^2 z(\alpha - s\beta + st\beta)(t-1) ds dt}{|\Delta z|^2} \\
&+ \frac{\beta^2 \partial_\alpha^\perp z(\alpha) \int_0^1 \int_0^1 \partial_\alpha^2 z(\alpha - \beta + t\beta + s\beta - ts\beta)(1-t) ds dt \cdot \int_0^1 (\partial_\alpha z(\alpha) + \partial_\alpha z(\alpha-\beta+t\beta)) dt}{|\partial_\alpha z(\alpha)|^2 |\Delta z|^2},
\end{aligned}
$$

therefore

$$
I_2^{21} \leq C \|\mathcal{F}(z)\|_{L^\infty} \|z\|_{\mathcal{C}^2}^2 \|u\|_{L^2}.
$$

For the term I_2^{22}

$$
\begin{aligned}
I_2^{22} = &-\frac{1}{\pi} PV \int_{\mathbb{R}} \frac{\partial_\alpha^\perp z(\alpha)}{|\partial_\alpha z(\alpha)|^2} \cdot \frac{\partial_\alpha^2 z(\alpha-\beta) - \partial_\alpha^2 z(\alpha)}{\beta} u(\alpha-\beta) d\beta \\
&- \frac{1}{\pi} PV \int_{\mathbb{R}} \frac{\partial_\alpha^\perp z(\alpha)}{|\partial_\alpha z(\alpha)|^2} \cdot \partial_\alpha^2 z(\alpha) \frac{u(\alpha-\beta)}{\beta} d\beta \\
&\leq C \|\mathcal{F}(z)\|_{L^\infty}^{\frac{1}{2}} \|z\|_{\mathcal{C}^{2,\delta}} \|u\|_{L^2} - \frac{1}{\pi} \frac{\partial_\alpha^\perp z(\alpha) \cdot \partial_\alpha^2 z(\alpha)}{|\partial_\alpha z(\alpha)|^2} H(u) \\
&\leq C \|\mathcal{F}(z)\|_{L^\infty}^{\frac{1}{2}} \|z\|_{\mathcal{C}^{2,\delta}} \|u\|_{L^2}.
\end{aligned}
$$

We can see easily for $\phi = \alpha - \beta + t\beta$

$$
\begin{aligned}
I_2^3 &= \frac{2}{\pi} PV \int_{\mathbb{R}} \frac{\beta^2 \int_0^1 \partial_\alpha^\perp z(\phi) dt \cdot \partial_\alpha z(\alpha-\beta) \int_0^1 \partial_\alpha z(\phi) dt \cdot \partial_\alpha z(\alpha-\beta)}{|\Delta z|^2} u(\alpha-\beta) d\beta \\
&\leq C \|\mathcal{F}(z)\|_{L^\infty} \|z\|_{\mathcal{C}^1}^2 \|z\|_{H^1}^2 \|u\|_{L^2}.
\end{aligned}
$$

Now, we consider

$$T_2^*(v) = -\frac{1}{\pi} PV \int_{\mathbb{R}} \frac{(h(\alpha)-z(\beta))^{\perp}\cdot\partial_\alpha z(\beta)}{|h(\alpha)-z(\beta)|^2} v(\beta) d\beta,$$

then

$$\begin{aligned}\partial_\alpha T_2^*(v) = & -\frac{1}{\pi} PV \int_{\mathbb{R}} \partial_\alpha\Big(\frac{(h(\alpha)-z(\alpha-\beta))^{\perp}\cdot\partial_\alpha z(\alpha-\beta)}{|h(\alpha)-z(\alpha-\beta)|^2}\Big) v(\alpha-\beta) d\beta \\ & - \frac{1}{\pi} PV \int_{\mathbb{R}} \frac{(h(\alpha)-z(\alpha-\beta))^{\perp}\cdot\partial_\alpha z(\alpha-\beta)}{|h(\alpha)-z(\alpha-\beta)|^2} \partial_\alpha v(\alpha-\beta) d\beta \equiv J_1 + J_2.\end{aligned}$$

Using $\partial_\alpha^{\perp} z\cdot\partial_\alpha z = 0$,

$$\begin{aligned} J_1 = & -\frac{1}{\pi} PV \int_{\mathbb{R}} \frac{\partial_\alpha^{\perp} h(\alpha)\cdot\partial_\alpha z(\alpha-\beta)}{|h(\alpha)-z(\alpha-\beta)|^2} v(\alpha-\beta) d\beta \\ & - \frac{1}{\pi} PV \int_{\mathbb{R}} \frac{(h(\alpha)-z(\alpha-\beta))^{\perp}\cdot\partial_\alpha^2 z(\alpha-\beta)}{|h(\alpha)-z(\alpha-\beta)|^2} v(\alpha-\beta) d\beta \\ & + \frac{2}{\pi} PV \int_{\mathbb{R}} \frac{(\Delta hz)^{\perp}\cdot\partial_\alpha z(\alpha-\beta)\Delta hz\cdot(\partial_\alpha h(\alpha)-\partial_\alpha z(\alpha-\beta))}{|h(\alpha)-z(\alpha-\beta)|^4} v(\alpha-\beta) d\beta \\ & \equiv J_1^1 + J_1^2 + J_1^3.\end{aligned}$$

Directly,

$$|J_1^1| \leq C\|d(z,h)\|_{L^\infty}\|z\|_{\mathcal{C}^2}\|h\|_{\mathcal{C}^1}\|v\|_{L^2},$$

$$|J_1^2| \leq C\|d(z,h)\|_{L^\infty}^{\frac{1}{2}}\|z\|_{\mathcal{C}^2}^2\|v\|_{L^2}$$

and

$$J_1^3 \leq C\|d(z,h)\|_{L^\infty}\|z\|_{\mathcal{C}^1}(\|z\|_{\mathcal{C}^1}+\|h\|_{\mathcal{C}^1})\|v\|_{L^2}.$$

Now, we study the term J_2. Since $\partial_\alpha z\cdot\partial_\alpha^{\perp} z = 0$,

$$\begin{aligned} J_2 = & -\frac{1}{\pi} PV \int_{\mathbb{R}} \partial_\beta\Big(\frac{(h(\alpha)-z(\alpha-\beta))^{\perp}\cdot\partial_\alpha z(\alpha-\beta)}{|h(\alpha)-z(\alpha-\beta)|^2}\Big) v(\alpha-\beta) d\beta \\ = & \frac{1}{\pi} PV \int_{\mathbb{R}} \frac{(h(\alpha)-z(\alpha-\beta))^{\perp}\cdot\partial_\alpha^2 z(\alpha-\beta)}{|h(\alpha)-z(\alpha-\beta)|^2} v(\alpha-\beta) d\beta \\ & - \frac{2}{\pi} PV \int_{\mathbb{R}} \frac{(\Delta zh)^{\perp}\cdot\partial_\alpha z(\alpha-\beta)\Delta zh\cdot\partial_\alpha z(\alpha-\beta)}{|h(\alpha)-z(\alpha-\beta)|^4} v(\alpha-\beta) d\beta.\end{aligned}$$

Using the same procedure as in term J_1,

$$|J_2^1| \leq C\|d(z,h)\|_{L^\infty}^{\frac{1}{2}}\|z\|_{\mathcal{C}^2}\|v\|_{L^2}$$

and

$$|J_2^2| \leq C\|d(z,h)\|_{L^\infty}\|z\|_{\mathcal{C}^1}^2\|v\|_{L^2}.$$

The operator $T_3^*(v)(\alpha)$ is estimated as well as $T_2^*(u)(\alpha)$ changing z with h and vice verse. For $T_4^*(v)(\alpha)$ we do the same as for $T_1^*(u)(\alpha)$ changing z for h and instead of $\mathcal{F}(z)$ the arc-chord condition for h, $\mathcal{F}(h)$. In conclusion,

$$\|\partial_\alpha \mathcal{T}^* w\|_{L^2} \leq C\|\mathcal{F}(z)\|_{L^\infty}^2\|\mathcal{F}(h)\|_{L^\infty}^2\|d(z,h)\|_{L^\infty}^2\|z\|_{\mathcal{C}^{2,\delta}}^2\|h\|_{\mathcal{C}^{2,\delta}}^2\|w\|_{L^2}.$$

■

Now it will be useful to consider the following functions: Let x be outside the curve $z(\alpha)$ and $h(\alpha)$, then we define

$$\begin{aligned} f_1(x) &= -\frac{1}{\pi}PV\int_{\mathbb{R}}\frac{(x-z(\beta))^\perp\cdot\partial_\alpha z(\beta)}{|x-z(\beta)|^2}u(\beta)d\beta \\ &= \frac{1}{\pi}PV\int_{\mathbb{R}}\frac{(x_2-z_2(\beta))\partial_\alpha z_1(\beta)}{|x-z(\beta)|^2}u(\beta)d\beta - \frac{1}{\pi}PV\int_{\mathbb{R}}\frac{(x_1-z_1(\beta))\partial_\alpha z_2(\beta)}{|x-z(\beta)|^2}u(\beta)d\beta. \end{aligned}$$

In the following we identify (u_1,u_2) with u_1+iu_2. Since $-u^\perp\cdot v = u_2v_1-u_1v_2$ and $(u_1+iu_2)(v_1+iv_2) = (u_1v_1+u_2v_2)+i(u_2v_1-u_1v_2)$ we get,

$$-u^\perp\cdot v = \Im(u\bar{v}).$$

Therefore, we can write

$$f_1(x) = \frac{1}{\pi}\Im\int_{\mathbb{R}}\frac{(x-z(\beta))\overline{\partial_\alpha z(\beta)}}{|x-z(\beta)|^2}u(\beta)d\beta$$

In the same way,

$$f_2(x) = \frac{1}{\pi}\Im\int_{\mathbb{R}}\frac{(x-h(\beta))\overline{\partial_\alpha h(\beta)}}{|x-h(\beta)|^2}v(\beta)d\beta$$

Both are the real part of the following Cauchy integrals

$$\begin{aligned} F_1(x) = f_1(x)+ig_1(x) &= \frac{1}{i\pi}\int_{\mathbb{R}}\frac{(x-z(\beta))\overline{\partial_\alpha z(\beta)}}{|x-z(\beta)|^2}u(\beta)d\beta, \\ F_2(x) = f_2(x)+ig_2(x) &= \frac{1}{i\pi}\int_{\mathbb{R}}\frac{(x-h(\beta))\overline{\partial_\alpha h(\beta)}}{|x-h(\beta)|^2}v(\beta)d\beta \end{aligned}$$

Taking $x = z(\alpha)+\epsilon\partial_\alpha^\perp z(\alpha)$ and letting $\epsilon\to 0$, we obtain

$$f_1(z(\alpha)) = T_1^*(u)(\alpha) - sign(\epsilon)u(\alpha). \tag{2.2}$$

and taking $x = h(\alpha)+\epsilon\partial_\alpha^\perp h(\alpha)$ and letting $\epsilon\to 0$

$$f_2(h(\alpha)) = T_4^*(v)(\alpha) - sign(\epsilon)v(\alpha). \tag{2.3}$$

Since the curve $z(\alpha)$ does not touch the curve $h(\alpha)$, we have

$$f_1(h(\alpha)) = T_2^*(u)(\alpha) \tag{2.4}$$

and

$$f_2(z(\alpha)) = T_3^*(v)(\alpha). \tag{2.5}$$

On the other hand,

$$\lim_{\epsilon\to 0} g_1(z(\alpha) \pm \epsilon\partial_\alpha^\perp z(\alpha)) = \lim_{\epsilon\to 0} \Im(F_1(z(\alpha) \pm \epsilon\partial_\alpha^\perp z(\alpha)) \equiv G_1(u)(\alpha)$$

where

$$G_1(u)(\alpha) = -\frac{1}{\pi} PV \int_{\mathbb{R}} \frac{(z(\alpha) - z(\beta)) \cdot \partial_\alpha z(\beta)}{|z(\alpha) - z(\beta)|^2} u(\beta) d\beta.$$

In the same way, taking limits

$$\lim_{\epsilon\to 0} g_2(h(\alpha) \pm \epsilon\partial_\alpha^\perp h(\alpha)) = \lim_{\epsilon\to 0} \Im(F_2(h(\alpha) \pm \epsilon\partial_\alpha^\perp h(\alpha)) \equiv G_2(u)(\alpha)$$

where

$$G_2(v)(\alpha) = -\frac{1}{\pi} PV \int_{\mathbb{R}} \frac{(h(\alpha) - h(\beta)) \cdot \partial_\alpha h(\beta)}{|h(\alpha) - h(\beta)|^2} v(\beta) d\beta$$

Therefore, we have the fact that $g_i^+(z(\alpha)) = g_i^-(z(\alpha))$ and $g_i^+(h(\alpha)) = g_i^-(h(\alpha))$ for $i = 1, 2$, where $(\cdot)^+$ denotes the limit obtained approaching from above to the boundaries in the normal direction and $(\cdot)^-$ from below.(This fact will be used on Subsection 2.1.3). Now we will show that $\mathcal{T}^* w = \lambda w \Rightarrow |\lambda| < 1$. If w is a eigenvector of $\mathcal{T}$, we have

$$\mathcal{T}^* w = \begin{pmatrix} T_1^* & T_3^* \\ T_2^* & T_4^* \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} T_1^* u + T_3^* v \\ T_2^* u + T_4^* v \end{pmatrix} = \begin{pmatrix} \lambda u \\ \lambda v \end{pmatrix} = \lambda w.$$

Let us compute ∇f_i for $i = 1, 2$. The identity

$$\begin{aligned} f_1(x) &= \frac{1}{\pi} \Im \int_{\mathbb{R}} \frac{(x - z(\beta))\overline{\partial_\alpha z(\beta)}}{|x - z(\beta)|^2} u(\beta) d\beta = \frac{1}{\pi} \Im \int_{\mathbb{R}} \frac{\overline{\partial_\alpha z(\beta)}}{\overline{(x - z(\beta))}} u(\beta) d\beta \\ &= -\frac{1}{\pi} \Im \int_{\mathbb{R}} \partial_\beta \ln(x - z(\beta)) u(\beta) d\beta = \frac{1}{\pi} \Im \int_{\mathbb{R}} \ln(x - z(\beta)) \partial_\beta u(\beta) d\beta \end{aligned}$$

yields

$$\nabla f_1(x) = \frac{1}{\pi} \Im \int_{\mathbb{R}} \partial_\beta u(\beta) \nabla \ln(x - z(\beta)) d\beta.$$

That is

$$\nabla f_1(x) = \frac{1}{\pi} \int_{\mathbb{R}} \partial_\beta u(\beta) \nabla \arg(x - z(\beta)) d\beta = \frac{1}{\pi} \int_{\mathbb{R}} \frac{(x - z(\beta))^\perp}{|x - z(\beta)|^2} \partial_\beta u(\beta) d\beta.$$

In the same way,

$$\nabla f_2(x) = \frac{1}{\pi} \int_{\mathbb{R}} \frac{(x - h(\beta))^\perp}{|x - h(\beta)|^2} \partial_\beta v(\beta) d\beta.$$

Taking $x = z(\alpha) + \epsilon z(\alpha)$ and letting $\epsilon \to 0$ in ∇f_1 we have

$$\nabla f_1(z(\alpha)) = 2BR(\partial_\alpha u, z)_z - sign(\epsilon) \frac{\partial_\alpha u(\alpha) \partial_\alpha z(\alpha)}{2|\partial_\alpha z(\alpha)|^2}. \tag{2.6}$$

On the other hand, taking $x = h(\alpha) + \epsilon h(\alpha)$ on ∇f_2 and letting $\epsilon \to 0$,

$$\nabla f_2(h(\alpha)) = 2BR(\partial_\alpha v, h)_h - sign(\epsilon) \frac{\partial_\alpha v(\alpha) \partial_\alpha h(\alpha)}{2|\partial_\alpha h(\alpha)|^2}. \tag{2.7}$$

Obviously,

$$\nabla f_1(h(\alpha)) = 2BR(\partial_\alpha u, z)_h \tag{2.8}$$

and

$$\nabla f_2(z(\alpha)) = 2BR(\partial_\alpha v, h)_z. \tag{2.9}$$

Assuming now that $\mathcal{T}^* w = \lambda w$, Ω_1 is the domain placed above of the curve $z(\alpha)$, Ω_2 is the domain between $z(\alpha)$ and $h(\alpha)$ and Ω_3 is below of the curve $h(\alpha)$. The analyticity of F_i for $i = 1, 2$ allows us to obtain:

$$\begin{aligned}
0 < &\int_{\Omega_1} |F_1'(x) + F_2'(x)|^2 dx = 2\int_{\Omega_1} |(\nabla f_1(x) + \nabla f_2(x))|^2 dx \\
&= -2\int_{\Omega_1} \Delta(f_1(x) + f_2(x))(f_1(x) + f_2(x))dx \\
&- 2\int_{\mathbb{T}} (f_1^+(z(\alpha)) + f_2^+(z(\alpha)))(\nabla f_1^+(z(\alpha)) + \nabla f_2^+(z(\alpha))) \cdot \frac{\partial_\alpha^\perp z(\alpha)}{|\partial_\alpha z(\alpha)|} d\alpha \\
&= 2\int_{\mathbb{T}} (-T_1^*(u)(\alpha) + u(\alpha) - T_3^*(v)(\alpha))(2BR(\partial_\alpha u, z)_z + 2BR(\partial_\alpha v, h)_z) \cdot \frac{\partial_\alpha^\perp z(\alpha)}{|\partial_\alpha z(\alpha)|} d\alpha \\
&= \int_{\mathbb{T}} (u(\alpha) - \lambda u(\alpha))M(u, v, h, z)d\alpha = (1 - \lambda)\int_{\mathbb{T}} u(\alpha)M(u, v, h, z)d\alpha \\
&\equiv (1 - \lambda)A,
\end{aligned} \tag{2.10}$$

$$\begin{aligned}
0 < &\int_{\Omega_2} |F_1'(x) + F_2'(x)|^2 dx \\
&= 2\int_{\mathbb{T}} (f_1(z(\alpha)) + f_2(z(\alpha)))(\nabla f_1(z(\alpha)) + \nabla f_2(z(\alpha))) \cdot \frac{\partial_\alpha^\perp z(\alpha)}{|\partial_\alpha z(\alpha)|} d\alpha \\
&- 2\int_{\mathbb{T}} (f_1^+(h(\alpha)) + f_2^+(h(\alpha)))(\nabla f_1^+(h(\alpha)) + \nabla f_2^+(h(\alpha))) \cdot \frac{\partial_\alpha^\perp h(\alpha)}{|\partial_\alpha h(\alpha)|} d\alpha \\
&= 2\int_{\mathbb{T}} (u(\alpha) + T_1^*(u)(\alpha) + T_3^*(v)(\alpha))M(u, v, h, z)d\alpha \\
&+ 2\int_{\mathbb{T}} (v(\alpha) - T_2^*(u)(\alpha) - T_4^*(v)(\alpha))(2BR(\partial_\alpha u, z)_h + 2BR(\partial_\alpha v, h)_h) \cdot \frac{\partial_\alpha^\perp z(\alpha)}{|\partial_\alpha z(\alpha)|} d\alpha \\
&= \int_{\mathbb{T}} (u(\alpha) + \lambda u(\alpha))M(u, v, h, z)d\alpha + 2\int_{\mathbb{T}} (v(\alpha) - \lambda v(\alpha))N(u, v, h, z) \\
&= (1 + \lambda)\int_{\mathbb{T}} u(\alpha)M(u, v, h, z)d\alpha + (1 - \lambda)\int_{\mathbb{T}} v(\alpha)N(u, v, h, z)d\alpha \\
&\equiv (1 + \lambda)A + (1 - \lambda)B
\end{aligned} \tag{2.11}$$

and

$$\begin{aligned}
0 < &\int_{\Omega_3} |F_1'(x) + F_2'(x)|^2 dx \\
&= 2\int_{\mathbb{T}} (f_1^-(h(\alpha)) + f_2^-(h(\alpha)))(\nabla f_1^-(h(\alpha)) + \nabla f_2^-(h(\alpha))) \cdot \frac{\partial_\alpha^\perp h(\alpha)}{|\partial_\alpha h(\alpha)|} d\alpha \\
&= 2\int_{\mathbb{T}} (v(\alpha) + T_2^*(u)(\alpha) + T_4^*(v)(\alpha))N(u, v, h, z)d\alpha \\
&= \int_{\mathbb{T}} (v(\alpha) + \lambda v(\alpha))N(u, v, h, z)d\alpha = (1 + \lambda)B
\end{aligned} \tag{2.12}$$

where we have used (2.2)-(2.9). Suppose that $|\lambda| \geq 1$ then $\lambda \in (-\infty, -1] \cup [1, \infty)$:

→ If $\lambda \in (-\infty, -1]$ then

i) For (2.10) we get that $A > 0$.

ii) For (2.12) we get that $B < 0$ and $\lambda \neq -1$.

iii) Therefore, (2.11) is a contradiction.

→ If $\lambda \in [1, \infty)$

i) For (2.10) we get that $A < 0$ and $\lambda \neq 1$.

ii) For (2.12) we get that $B > 0$.

iii) Therefore, (2.11) is a contradiction.

Thus $|\lambda| < 1$. At this point, since $\mathcal{T}^*$ is a compact operator, we know that there exists $(I - M\mathcal{T}^*)^{-1}$ for $M = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}$ with $|m_i| < 1$ for $i = 1, 2$. Our propose is to prove that $H^{\frac{1}{2}}$-norm of the inverse operator are bounded by $\exp(C|||z, h|||^2)$ where $|||z, h|||^2 = \|\mathcal{F}(z)\|^2_{L^\infty} + \|d(z, h)\|^2_{L^\infty} + \|z\|^2_{H^3}$. To prove that we will start with the following proposition:

Proposition 2.1.2. *The norms* $\|(I \pm \mathcal{T}^*)^{-1}\|_{L^2_0}$ *are bounded by* $\exp(C|||z, h|||^2)$ *for some universal constant* C. *Here the space* L^2_0 *is the usual* L^2 *with the extra condition of mean value zero.*

Proof. The proof follows if we demonstrate the estimate

$$e^{-C|||z,h|||^2} \leq \frac{\|\varpi - \mathcal{T}^*\varpi\|_{L^2_0}}{\|\varpi + \mathcal{T}^*\varpi\|_{L^2_0}} \leq e^{C|||z,h|||^2} \tag{2.13}$$

valid for every nonzero $\varpi \in L^2_0 \times L^2_0$. This is because if we assume $\|\varpi - \mathcal{T}^*\varpi\|_{L^2_0} \leq e^{-2C|||z,h|||^2}$ for some $\|\varpi\|_{L^2_0} = 1$ then we obtain $\|\varpi + \mathcal{T}^*\varpi\|_{L^2_0} \geq 2\|\varpi\|_{L^2_0} - e^{-2C|||z,h|||^2} \geq 1$ which contradicts 2.13. Therefore we must have $\|\varpi - \mathcal{T}^*\varpi\|_{L^2_0} \geq e^{-2C|||z,h|||^2}$ for all $\|\varpi\|_{L^2_0} = 1$ i.e. $\|(I - \mathcal{T}^*)^{-1}\|_{L^2_0} \leq e^{2C|||z,h|||^2}$. Similarly we also have $\|(I + \mathcal{T}^*)^{-1}\|_{L^2_0} \leq e^{2C|||z,h|||^2}$. Since

$$\varpi + \mathcal{T}^*\varpi = \begin{pmatrix} u + T_1^*u + T_3^*v \\ v + T_2^*u + T_4^*v \end{pmatrix} = \begin{pmatrix} f_1^-(z(\alpha)) + f_2(z(\alpha)) \\ f_1(h(\alpha)) + f_2^-(h(\alpha)) \end{pmatrix} \equiv \begin{pmatrix} m^+ \\ w \end{pmatrix}$$

$$\varpi - \mathcal{T}^*\varpi = \begin{pmatrix} u - T_1^*u - T_3^*v \\ v - T_2^*u - T_4^*v \end{pmatrix} = (-1)\begin{pmatrix} f_1^+(z(\alpha)) + f_2(z(\alpha)) \\ f_1(h(\alpha)) + f_2^+(h(\alpha)) \end{pmatrix} \equiv (-1)\begin{pmatrix} f \\ m^- \end{pmatrix}$$

Next we will see that we can write the above function as some operators, which we call $\mathcal{H}_i$ for $i = 1, 2, 3$, where i denotes the corresponding domain Ω_1, Ω_2, and Ω_3(See Subsection 2.1.3). The relations with these operator are:

$$\begin{aligned} m^+ &= \mathcal{H}_2^z(f, m^-), \\ w &= \mathcal{H}_3(f, m^-), \\ f &= \mathcal{H}_1(m^+, w), \\ m^- &= \mathcal{H}_2^h(m^+, w). \end{aligned}$$

And we will prove that

$$\|\mathcal{H}_i(\varpi)\|_{L^2} \leq e^{C|||z,h|||^2}\|\varpi\|_{L^2},$$

where C denotes a universal constant not necessarily the same at each occurrence. With all these assumptions, the proof is as follows:

$$\begin{aligned}\|\varpi + \mathcal{T}^*\varpi\|_{L^2_0} &= \|\begin{pmatrix}\mathcal{H}_2^z(f,m^-)\\ \mathcal{H}_3(f,m^-)\end{pmatrix}\|_{L^2_0} \leq e^{C|||z,h|||^2}\|\begin{pmatrix}f\\ m^-\end{pmatrix}\|_{L^2_0}\\ &= e^{2C|||z,h|||^2}\|\varpi - \mathcal{T}^*\varpi\|_{L^2_0}.\end{aligned}$$

In the same way,

$$\begin{aligned}\|\varpi - \mathcal{T}^*\varpi\|_{L^2_0} &= \|\begin{pmatrix}\mathcal{H}_1(m^+,w)\\ \mathcal{H}_2^h(m^+,w)\end{pmatrix}\|_{L^2_0} \leq e^{C|||z,h|||^2}\|\begin{pmatrix}m^+\\ w\end{pmatrix}\|_{L^2_0}\\ &= e^{2C|||z,h|||^2}\|\varpi + \mathcal{T}^*\varpi\|_{L^2_0}.\end{aligned}$$

■

Once we have the estimation of $(I \pm \mathcal{T}^*)^{-1}$, we introduce the term $M = \begin{pmatrix}m_1 & 0\\ 0 & m_2\end{pmatrix}$ with $|m_i| < 1$ for all $i = 1, 2$.

Lemma 2.1.2. *The following estimate holds:*

$$\|(I + M\mathcal{T}^*)^{-1}\|_{L^2_0} \leq e^{C|||z,h|||^2}$$

for a universal constant C and $|m_i| \leq 1$ for $i = 1, 2$.

Proof. If we look at the identity $I+M\mathcal{T}^* = M(I+\mathcal{T}^*)+(I-MI)$, using the estimate on proposition 2.1.2 we can conclude that

$$\|(I + M\mathcal{T}^*)^{-1}\|_{L^2_0} \leq \exp C|||z,h|||^2$$

for $1 - e^{-C_1|||z,h|||^2} \leq |m_i| \leq 1$. For $|m_i| \leq 1 - e^{-C_1|||z,h|||^2}$: Since $\|M\mathcal{T}^*\|_{L^2} < 1$ then we can write $(I + M\mathcal{T}^*)^{-1} = \sum_n (M\mathcal{T}^*)^n$. Taking norms,

$$\|(I + M\mathcal{T}^*)^{-1}\|_{L^2_0} \leq \sum_n \|M\mathcal{T}^*\|^n_{L^2_0} \leq \sum_n (1 - e^{-C_1|||z,h|||^2})^n = e^{C_1|||z,h|||^2}$$

■

Now we are in position to prove the $H^{\frac{1}{2}}$-norm,

Proposition 2.1.3. *For $m_i \leq 1$ the following estimate holds*

$$\|(I + M\mathcal{T})^{-1}\|_{H^{\frac{1}{2}}_0} = \|(I + M\mathcal{T}^*)^{-1}\|_{H^{\frac{1}{2}}_0} \leq e^{C|||z,h|||^2},$$

where C is a universal constant and $M = \begin{pmatrix}m_1 & 0\\ 0 & m_2\end{pmatrix}$.

Proof. We will use the same idea as in Proposition 2.1.2, therefore we are going to prove:

$$e^{-C|||z,h|||^2} \leq \frac{\|\varpi - M\mathcal{T}^*\varpi\|_{H_0^{\frac{1}{2}}}}{\|\varpi + M\mathcal{T}^*\varpi\|_{H_0^{\frac{1}{2}}}} \leq e^{C|||z,h|||^2}.$$

To do that, using proposition 2.1.1 and $|\mu_i| < 1$,

$$\begin{aligned}
&\|\Lambda^{\frac{1}{2}}(\varpi + M\mathcal{T}^*\varpi)\|_{L_0^2} \leq \|\Lambda^{\frac{1}{2}}(\varpi - M\mathcal{T}^*\varpi)\|_{L_0^2} + 2\|M\Lambda^{\frac{1}{2}}(\mathcal{T}^*\varpi)\|_{L_0^2} \\
&\leq \|\Lambda^{\frac{1}{2}}(\varpi - M\mathcal{T}^*\varpi)\|_{L_0^2} + 2\|\mathcal{T}^*\varpi\|_{H^1} \\
&\leq \|\Lambda^{\frac{1}{2}}(\varpi - M\mathcal{T}^*\varpi)\|_{L_0^2} + e^{C|||z,h|||^2}\|\varpi\|_{L_0^2}.
\end{aligned}$$

Using the estimate of Lemma 2.1.2,

$$\|\varpi\|_{L_0^2} = \|(I - M\mathcal{T}^*)^{-1}(I - M\mathcal{T}^*)\varpi\|_{L_0^2} \leq e^{C|||z,h|||^2}\|\varpi - M\mathcal{T}^*\varpi\|_{L_0^2}.$$

Therefore,

$$\|\varpi + M\mathcal{T}^*\varpi\|_{H_0^{\frac{1}{2}}} \leq e^{C|||z,h|||^2}\|\varpi - M\mathcal{T}^*\varpi\|_{H_0^{\frac{1}{2}}}$$

Analogously, we get

$$\|\varpi - M\mathcal{T}^*\varpi\|_{H_0^{\frac{1}{2}}} \leq e^{C|||z,h|||^2}\|\varpi + M\mathcal{T}^*\varpi\|_{H_0^{\frac{1}{2}}}$$

and we finish the proof. ■

2.1.3 $\mathcal{H}_i$ operators

The truth of the above results depend on the existence of the $\mathcal{H}_i$ operators which we have denoted on Proposition 2.1.2. We will start with considering a flat domain, where the boundaries are $(x, 0)$ and $(x, 1)$. Let be F a harmonic function, decaying at infinity, above $(x, 1)$ such that

$$\begin{cases} \Delta F = 0 \\ F(x,1) = f(x) \end{cases}$$

Taking Fourier transform, we can get $\hat{F}(\xi, y) = e^{-|\xi|(y-1)}\hat{f}(\xi)$. Now, if we calculate the harmonic conjugate,which we will call G, we can get $\hat{G}(\xi, y) = -isign(\xi)\hat{f}(\xi)e^{-|\xi|(y-1)}$. And therefore, $\hat{G}(\xi, 1) = -isign(\xi)\hat{f}(\xi)$. Now we consider between the boundaries the harmonic function M such that,

$$\begin{cases} \Delta M = 0 \\ M(x,1) = m^+(x) \\ M(x,0) = m^-(x) \end{cases}$$

Taking Fourier Transform and computing the harmonic conjugate, we get
$\hat{N}(\xi, y) = iA\cosh(\xi y) + iB\sinh(\xi y)$. At the end, we want to relate these harmonic function with ours F_i described at the Subsection 2.1.2. We saw that $g_i^+(z(\alpha)) = g_i^-(z(\alpha))$ and $g_i^+(h(\alpha)) = g_i^-(h(\alpha))$ for $i = 1, 2$. That is why we consider now $G(x, 1) = N(x, 1)$ and before $N(x, 0) = R(x, 0)$. Therefore, since

$$\hat{N}(\xi, 1) = iA\cosh(\xi) + iB\sinh(\xi) = \hat{G}(\xi, 1) = -isign(\xi)\hat{f}(\xi),$$

$$\hat{M}(\xi,0) = B = \hat{m}^-(\xi)$$

then

$$A = \frac{sign(\xi)\hat{f}(\xi) - \hat{m}^-(\xi)\sinh(\xi)}{\cosh(\xi)}$$

and

$$\hat{m}^+(\xi) = \hat{M}(\xi,1) = \frac{sign(\xi)\hat{f}(\xi)\sinh(\xi) + \hat{m}^-(\xi)}{\cosh(\xi)}.$$

Moreover,

$$\hat{N}(\xi,0) = \frac{isign(\xi)\hat{f}(\xi) - i\hat{m}^-(\xi)\sinh(\xi)}{\cosh(\xi)}.$$

Finally, we consider an harmonic function W below $(x,0)$ in such a way that

$$\begin{cases} \Delta W = 0 \\ W(x,0) = w(x) \end{cases}$$

With the same procedure as before, we get the harmonic conjugate $\hat{R}(\xi,y) = isign(\xi)\hat{w}(\xi)e^{|x|y}$. Since, $\hat{R}(\xi,0) = \hat{N}(\xi,0)$ therefore

$$\hat{w}(\xi) = \frac{\hat{f}(\xi) - sign(\xi)\hat{m}^-(\xi)\sinh(\xi)}{\cosh(\xi)}.$$

Thus we just put $\hat{m}^+$ and $\hat{w}$ as a function of $\hat{f}$ and $\hat{m}^-$. We do this going from the top of the domain to the bottom. If we do the same going from the bottom to the top we will obtain

$$\begin{aligned} \hat{f}(\xi) &= \frac{-\hat{w}(\xi) - sign(\xi)\hat{m}^+(\xi)\sinh(\xi)}{\cosh(\xi)}, \\ \hat{m}^-(\xi) &= \frac{\hat{m}^+(\xi) - sign(\xi)\hat{w}(\xi)\sinh(\xi)}{\cosh(\xi)}. \end{aligned}$$

Here we define ours operators like:

$$\begin{aligned} \widehat{H_1(m^+,w)} &= \frac{-\hat{w}(\xi) - sign(\xi)\hat{m}^+(\xi)\sinh(\xi)}{\cosh(\xi)}, \\ \widehat{H_2^h(m^+,w)} &= \frac{\hat{m}^+(\xi) - sign(\xi)\hat{w}(\xi)\sinh(\xi)}{\cosh(\xi)}, \\ \widehat{H_2^z(f,m^-)} &= \frac{\hat{m}^-(\xi) + sign(\xi)\hat{f}(\xi)\sinh(\xi)}{\cosh(\xi)}, \\ \widehat{H_3(f,m^-)} &= \frac{\hat{f}(\xi) - sign(\xi)\hat{f}(\xi)\sinh(\xi)}{\cosh(\xi)}, \end{aligned}$$

which are bounded in L^2.

Let ϕ_i the conformal mapping from the Ω_i domain to the "flat" domain (See the figure 2.1), then the corresponding operator in the "curved" domain are denoted by $\mathcal{H}_i$. For the L^2-norm of the $\mathcal{H}_i$ operator we can repeat the proofs in [11] for their Hilbert operator $\mathcal{H}_1$. To do this we only have to look at the formulas:

$$\begin{aligned} \mathcal{H}_1(m^+,w) &= H_1(m^+\circ\phi_1^{-1}, w\circ\phi_1^{-1})\circ\phi_1, \\ \mathcal{H}_2^h(m^+,w) &= H_2^h(m^+\circ\phi_2^{-1}, w\circ\phi_2^{-1})\circ\phi_2, \end{aligned}$$

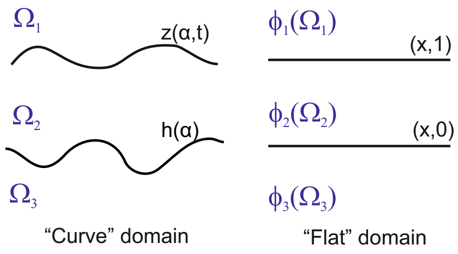


Figure 2.1: Conformal maps ϕ_i

$$\mathcal{H}_2^z(f,m^-) = H_2^z(f\circ\phi_2^{-1}, m^-\circ\phi_2^{-1})\circ\phi_2,$$
$$\mathcal{H}_3(f,m^-) = H_3(f\circ\phi_3^{-1}, m^-\circ\phi_3^{-1})\circ\phi_3.$$

Since our parametric curves $z(\alpha)$ and $h(\alpha)$ are $\mathcal{C}^{2,\delta}$ satisfying the arc-chord conditions $\|\mathcal{F}(z)\|_{L^\infty}<\infty$, $\|\mathcal{F}(h)\|_{L^\infty}<\infty$ and the distance $\|d(z,h)\|_{L^\infty}<\infty$. Then we have tangent balls to the boundary contained inside the domains Ω_i. Furthermore, we can estimate from below the radius of those balls by $C|||z,h|||^{-1}$(As in Lemma 4.3 in [11]). Following the steps of the proof of Lemma 4.4 in [11] we can conclude that

$$\|\mathcal{H}_i\|_{L^2}\le e^{C|||z,h|||^2}$$

for all $i=1,2,3$.

2.2 Estimates on ϖ

In this section we show that the norm of amplitude of the vorticity $\varpi=(\varpi_1,\varpi_2)$ is bounded in H^k, for $k\ge 2$.

Lemma 2.2.1. *Let $\varpi=(\varpi_1,\varpi_2)$ be a function given by*

$$\varpi_1(\alpha) = -\lambda_1 T_1(\varpi_1)(\alpha)-\lambda_1 T_2(\varpi_2)(\alpha)-N\partial_\alpha z_2(\alpha), \tag{2.14}$$
$$\varpi_2(\alpha) = -\lambda_2 T_3(\varpi_1)(\alpha)-\lambda_2 T_4(\varpi_2)(\alpha) \tag{2.15}$$

where $\lambda_1=\frac{\mu_2-\mu_1}{\mu_1+\mu_2}$, $\lambda_2=\frac{\kappa_1-\kappa_2}{\kappa_1+\kappa_2}$ and $N=2\kappa_1 g\frac{\rho_2-\rho_1}{\mu_2+\mu_1}$.

Then

$$\|\varpi\|_{H^k}\le \exp C(\|\mathcal{F}(z)\|^2_{L^\infty}+\|d(z,h)\|^2_{L^\infty}+\|z\|^2_{H^{k+1}}) \tag{2.16}$$

for $k\ge 2$.

Proof. We can write,

$$\varpi = M\mathcal{T}\varpi - v \tag{2.17}$$

where $M=\begin{pmatrix}-\lambda_1 & 0\\ 0 & -\lambda_2\end{pmatrix}$, $\mathcal{T}=\begin{pmatrix}T_1 & T_2\\ T_3 & T_4\end{pmatrix}$ and $v=\begin{pmatrix}N\partial_\alpha z_2(\alpha)\\ 0\end{pmatrix}$. The formula (2.17) is equivalent to

$$\varpi=(I+M\mathcal{T})^{-1}v.$$

It yields

$$\|\varpi\|_{H^{\frac12}}\le\|(I+M\mathcal{T})^{-1}\|_{H^{\frac12}}\|\partial_\alpha z_2\|_{H^{\frac12}}.$$

Since $|\lambda_i| < 1$ for all i, the proposition 2.1.3 gives

$$\|\varpi\|_{H^{\frac{1}{2}}} \leq e^{C|||z,h|||^2}.$$

Recall that $|||z,h|||^2 = \|\mathcal{F}(z)\|^2_{L^\infty} + \|d(z,h)\|^2_{L^\infty} + \|z\|^2_{H^3}$. Now, we consider the H^{k+1}-norm

$$\|\varpi\|_{H^{k+1}} = \|\varpi_1\|_{H^{k+1}} + \|\varpi_2\|_{H^{k+1}}.$$

Then, we study each component one by one.

Taking the k derivative of (2.14) we get:

$$\partial^k_\alpha \varpi_1(\alpha) = -\lambda_1 \partial^k_\alpha (2BR(\varpi_1, z)_z \cdot \partial_\alpha z(\alpha)) - \lambda_1 \partial^k_\alpha (2BR(\varpi_2, h)_z \cdot \partial_\alpha z(\alpha)) - N\partial^{k+1}_\alpha z_2(\alpha).$$

Using Leibniz's rule we have,

$$\begin{aligned}
2\partial^k_\alpha (BR(\varpi_1, z)_z \cdot \partial_\alpha z(\alpha)) &= \sum_{j=0}^{k} \frac{C_k}{\pi} \int_{\mathbb{R}} \partial^{k-j}_\alpha \Big(\frac{\Delta z^\perp \cdot \partial_\alpha z(\alpha)}{|\Delta z|^2}\Big) \partial^j_\alpha \varpi_1(\alpha - \beta) d\beta \\
= \sum_{j=0}^{k-1} \frac{C_k}{\pi} \int_{\mathbb{R}} \partial^{k-j}_\alpha \Big(\frac{\Delta z^\perp \cdot \partial_\alpha z(\alpha)}{|\Delta z|^2}\Big) \partial^j_\alpha \varpi_1(\alpha - \beta) d\beta + T_1(\partial^k_\alpha \varpi_1)(\alpha)
\end{aligned}$$

and

$$2\partial^k_\alpha (BR(\varpi_2, h)_z \cdot \partial_\alpha z(\alpha)) = \sum_{j=0}^{k-1} \frac{C_k}{\pi} \int_{\mathbb{R}} \partial^{k-j}_\alpha \Big(\frac{\Delta zh^\perp \cdot \partial_\alpha z(\alpha)}{|\Delta zh|^2}\Big) \partial^j_\alpha \varpi_2(\alpha - \beta) d\beta + T_2(\partial^k_\alpha \varpi_2)(\alpha).$$

Recall that $\Delta z = z(\alpha) - z(\alpha - \beta)$ and $\Delta zh = z(\alpha) - h(\alpha - \beta)$. Therefore, we obtain

$$\partial^k_\alpha \varpi_1(\alpha) + \lambda_1 T_1(\partial^k_\alpha \varpi_1)(\alpha) + \lambda_1 T_2(\partial^k_\alpha \varpi_2)(\alpha) = R^1_k(\varpi_1) + R^2_k(\varpi_2) - N\partial^{k+1}_\alpha z_2(\alpha)$$

where

$$\begin{aligned}
R^1_k(\varpi_1) &= \sum_{j=0}^{k-1} \frac{C_k}{\pi} \int_{\mathbb{R}} \partial^{k-j}_\alpha \Big(\frac{\Delta z^\perp \cdot \partial_\alpha z(\alpha)}{|\Delta z|^2}\Big) \partial^j_\alpha \varpi_1(\alpha - \beta) d\beta, \\
R^2_k(\varpi_2) &= \sum_{j=0}^{k-1} \frac{C_k}{\pi} \int_{\mathbb{R}} \partial^{k-j}_\alpha \Big(\frac{\Delta zh^\perp \cdot \partial_\alpha z(\alpha)}{|\Delta zh|^2}\Big) \partial^j_\alpha \varpi_2(\alpha - \beta) d\beta.
\end{aligned}$$

If we observe that

$$\partial_\alpha T_1(\varpi_1)(\alpha) = \frac{1}{\pi} \int_{\mathbb{R}} \partial_\alpha \Big(\frac{\Delta z^\perp \cdot \partial_\alpha z(\alpha)}{|\Delta z|^2}\Big) \varpi_1(\alpha - \beta) d\beta + T_1(\partial_\alpha \varpi_1)(\alpha)$$

we get

$$\begin{aligned}
R^1_k(\varpi_1) &= \partial^{k-1}_\alpha \Big(\frac{1}{\pi} \int_{\mathbb{R}} \partial_\alpha \Big(\frac{\Delta z^\perp \cdot \partial_\alpha z(\alpha)}{|\Delta z|^2}\Big) \varpi_1(\alpha - \beta) d\beta\Big) \\
&= \partial^{k-1}_\alpha (\partial_\alpha T_1(\varpi_1)(\alpha) - T_1(\partial_\alpha \varpi_1)(\alpha)) = \partial^k_\alpha T_1(\varpi_1)(\alpha) - \partial^{k-1}_\alpha T_1(\partial_\alpha \varpi_1)(\alpha)
\end{aligned}$$

and

$$R_k^2(\varpi_2) = \partial_\alpha^k T_2(\varpi_2)(\alpha) - \partial_\alpha^{k-1} T_2(\partial_\alpha \varpi_2)(\alpha).$$

Taking the k-derivatives in (2.15), we get

$$\partial_\alpha^k \varpi_2(\alpha) + \lambda_2 T_3(\partial_\alpha^k \varpi_1)(\alpha) + \lambda_2 T_4(\partial_\alpha^k \varpi_2)(\alpha) = R_k^3(\varpi_1) + R_k^4(\varpi 2)$$

with

$$R_k^3(\varpi_1) = \partial_\alpha^k T_3(\varpi_1)(\alpha) - \partial_\alpha^{k-1} T_3(\partial_\alpha \varpi_1)(\alpha),$$
$$R_k^4(\varpi_2) = \partial_\alpha^k T_4(\varpi_2)(\alpha) - \partial_\alpha^{k-1} T_4(\partial_\alpha \varpi_2)(\alpha).$$

Next let us consider

$$\begin{aligned} &\Lambda^{\frac{1}{2}}\partial_\alpha^k \varpi_1(\alpha) + \lambda_1 T_1(\Lambda^{\frac{1}{2}}\partial_\alpha^k \varpi_1)(\alpha) + \lambda_1 T_2(\Lambda^{\frac{1}{2}}\partial_\alpha^k \varpi_2)(\alpha) \\ &= \lambda_1 T_1(\Lambda^{\frac{1}{2}}\partial_\alpha^k \varpi_1)(\alpha) + \lambda_1 T_2(\Lambda^{\frac{1}{2}}\partial_\alpha^k \varpi_2)(\alpha) - \lambda_1 \Lambda^{\frac{1}{2}} T_1(\partial_\alpha^k \varpi_1)(\alpha) - \lambda_1 \Lambda^{\frac{1}{2}} T_2(\partial_\alpha^k \varpi_2)(\alpha) \\ &+ \Lambda^{\frac{1}{2}} R_k^1(\varpi_1)(\alpha) + \Lambda^{\frac{1}{2}} R_k^2(\varpi_2)(\alpha) - N\Lambda^{\frac{1}{2}}\partial_\alpha^{k+1} z_2(\alpha) \end{aligned} \tag{2.18}$$

and

$$\begin{aligned} &\Lambda^{\frac{1}{2}}\partial_\alpha^k \varpi_2(\alpha) + \lambda_2 T_3(\Lambda^{\frac{1}{2}}\partial_\alpha^k \varpi_1)(\alpha) + \lambda_2 T_4(\Lambda^{\frac{1}{2}}\partial_\alpha^k \varpi_2)(\alpha) \\ &= \lambda_2 T_3(\Lambda^{\frac{1}{2}}\partial_\alpha^k \varpi_1)(\alpha) + \lambda_2 T_4(\Lambda^{\frac{1}{2}}\partial_\alpha^k \varpi_2)(\alpha) - \lambda_2 \Lambda^{\frac{1}{2}} T_3(\partial_\alpha^k \varpi_1)(\alpha) - \lambda_2 \Lambda^{\frac{1}{2}} T_4(\partial_\alpha^k \varpi_2)(\alpha) \\ &+ \Lambda^{\frac{1}{2}} R_k^3(\varpi_1)(\alpha) + \Lambda^{\frac{1}{2}} R_k^4(\varpi_2)(\alpha). \end{aligned} \tag{2.19}$$

Then, we write

$$\begin{pmatrix} \Lambda^{\frac{1}{2}}\partial_\alpha^k \varpi_1 \\ \Lambda^{\frac{1}{2}}\partial_\alpha^k \varpi_2 \end{pmatrix} + \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}\begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix}\begin{pmatrix} \Lambda^{\frac{1}{2}}\partial_\alpha^k \varpi_1 \\ \Lambda^{\frac{1}{2}}\partial_\alpha^k \varpi_2 \end{pmatrix} = \begin{pmatrix} S_1 \\ S_2 \end{pmatrix}$$

where S_1 is the right hand side of (2.18) and S_2 the right hand side of (2.19). Using the estimate for the inverse $(I + M\mathcal{T})^{-1}$ in the space $H^{\frac{1}{2}}$ we get

$$\|\varpi\|_{H^{k+1}} \leq \|\Lambda^{\frac{1}{2}}\partial_\alpha^k \varpi\|_{H^{\frac{1}{2}}} \leq C\|(I + M\mathcal{T})^{-1}\|_{H^{\frac{1}{2}}} \|S\|_{H^{\frac{1}{2}}} \leq e^{C|||z,h|||^2} \|S\|_{H^{\frac{1}{2}}}.$$

We have that

$$S = \begin{pmatrix} S_1 \\ S_2 \end{pmatrix} = M\mathcal{T}(\Lambda^{\frac{1}{2}}\partial_\alpha^k \varpi) - M\Lambda^{\frac{1}{2}}(\mathcal{T}(\partial_\alpha^k \varpi)) + \Lambda^{\frac{1}{2}}(\mathbb{R}_k(\varpi)) - \begin{pmatrix} N\Lambda^{\frac{1}{2}}\partial_\alpha^{k+1} z_2(\alpha) \\ 0 \end{pmatrix}$$

where $\mathbb{R}_k(\varpi) = \begin{pmatrix} R_k^1 & R_k^2 \\ R_k^3 & R_k^4 \end{pmatrix}\begin{pmatrix} \varpi_1 \\ \varpi_2 \end{pmatrix}$. Thus,

$$\|S\|_{H^{\frac{1}{2}}} \leq C\|\mathcal{T}(\Lambda^{\frac{1}{2}}\partial_\alpha^k \varpi)\|_{H^{\frac{1}{2}}} + \|\mathcal{T}(\partial_\alpha^k \varpi))\|_{H^1} + \|\mathbb{R}_k(\varpi)\|_{H^1} + \|z\|_{H^{k+2}}.$$

Using the lemma 2.1.1,

$$\|\mathcal{T}(\partial_\alpha^k \varpi)\|_{H^1} \leq C\|\mathcal{F}(z)\|_{L^\infty}^2 \|\mathcal{F}(h)\|_{L^\infty}^2 \|d(z,h)\|_{L^\infty}^2 \|z\|_{\mathcal{C}^2}^4 \|h\|_{\mathcal{C}^2}^4 \|\varpi\|_{H^k}.$$

Since $R_k^i(\varpi_j) = \partial_\alpha^k T_i(\varpi_j)(\alpha) - \partial_\alpha^{k-1} T_i(\partial_\alpha \varpi_j)(\alpha)$ for $i, j = 1, 2, 3, 4$ we can write

$$\mathbb{R}_k(\varpi) = \partial_\alpha^k \mathcal{T}(\varpi) - \partial_\alpha^{k-1}\mathcal{T}(\partial_\alpha \varpi).$$

Then using the lemma 2.2.2 (proved below),

$$\begin{aligned}
&\|\mathbb{R}_k(\varpi)\|_{H^1} \le \|\partial_\alpha^k \mathcal{T}(\varpi)\|_{H^1} + \|\partial_\alpha^{k-1}\mathcal{T}(\partial_\alpha\varpi)\|_{H^1} \le \|\mathcal{T}(\varpi)\|_{H^{k+1}} + \|\mathcal{T}(\partial_\alpha \varpi)\|_{H^k} \\
&\le C|||z,h|||^2(\|z\|^2_{H^{k+2}} + \|h\|^2_{H^{k+2}})\|\varpi\|_{H^k} + C|||z,h|||^2(\|z\|^2_{H^{k+1}} + \|h\|^2_{H^{k+1}})\|\partial_\alpha\varpi\|_{H^{k-1}}.
\end{aligned}$$

Finally, using lemma 2.1.1

$$\begin{aligned}
&\|\mathcal{T}(\Lambda^{\frac{1}{2}}\partial_\alpha^k\varpi)\|_{H^{\frac{1}{2}}} \le \|\mathcal{T}(\Lambda^{\frac{1}{2}}\partial_\alpha^k\varpi)\|_{H^1} \\
&\le C\|\mathcal{F}(z)\|^2_{L^\infty}\|\mathcal{F}(h)\|^2_{L^\infty}\|d(z,h)\|^2_{L^\infty}\|z\|^4_{C^2}\|h\|^4_{C^2}\|\Lambda^{\frac{1}{2}}\partial_\alpha^k\varpi\|_{L^2}.
\end{aligned}$$

Since

$$\partial_\alpha^k\varpi + M\mathcal{T}\partial_\alpha^k\varpi = \mathbb{R}_k(\varpi) - \begin{pmatrix} N\partial_\alpha^{k+1}z_2 \\ 0 \end{pmatrix},$$

then

$$\partial_\alpha^k\varpi = (I + M\mathcal{T})^{-1}(\mathbb{R}_k(\varpi) - \begin{pmatrix} N\partial_\alpha^{k+1}z_2 \\ 0 \end{pmatrix}).$$

Therefore,

$$\begin{aligned}
&\|\Lambda^{\frac{1}{2}}\partial_\alpha^k\varpi\|_{L^2} = \|\partial_\alpha^k\varpi\|_{H^{\frac{1}{2}}} \le \|(I+M\mathcal{T})^{-1}\|_{H^{\frac{1}{2}}}(\|\mathbb{R}_k(\varpi)\|_{H^{\frac{1}{2}}} + \|z\|_{H^{k+\frac{3}{2}}}) \\
&\le e^{C|||z,h|||^2}(C|||z,h|||^2(\|z\|^2_{H^{k+2}} + \|h\|^2_{H^{k+2}})\|\varpi\|_{H^k} + \|z\|_{H^{k+\frac{3}{2}}}).
\end{aligned}$$

In conclusion,

$$\|\varpi\|_{H^{k+1}} \le e^{C|||z,h|||^2}(\|\mathcal{F}(z)\|^2_{L^\infty}\|d(z,h)\|^2_{L^\infty}\|z\|^2_{H^{k+2}}\|\varpi\|_{H^k} + \|z\|_{H^{k+2}}).$$

For $k = \frac{1}{2}$, since $\|\varpi\|_{H^{\frac{1}{2}}} \le e^{C|||z,h|||^2}$ then

$$\|\varpi\|_{H^{\frac{3}{2}}} \le \exp C(\|\mathcal{F}(z)\|^2_{L^\infty} + \|d(z,h)\|^2_{L^\infty} + \|z\|^2_{H^{\frac{5}{2}}}).$$

Therefore using induction on $k \ge 2$ allows us to finish the proof. ∎

Lemma 2.2.2. *The operator $\mathcal{T}$ maps Sobolev space $H^k \times H^k$, $k \ge 1$, into $H^{k+1} \times H^{k+1}$ as long as $z, h \in H^{k+2}$ and satisfies the estimate*

$$\|\mathcal{T}\|_{H^k\times H^k \to H^{k+1}\times H^{k+1}} \le C|||z,h|||^2\|z\|^2_{H^{k+2}}$$

Proof. For the lemma 5.2 in [11] we have

$$\|T_1(\varpi_1)\|_{H^{k+1}} \le C(\|\mathcal{F}(z)\|^2_{L^\infty} + \|z\|^2_{H^3})\|z\|^2_{H^{k+2}}\|\varpi_1\|_{H^k}$$

and changing z for h then

$$\|T_4(\varpi_2)\|_{H^{k+1}} \le C(\|\mathcal{F}(h)\|^2_{L^\infty} + \|h\|^2_{H^3})\|h\|^2_{H^{k+2}}\|\varpi_2\|_{H^k}.$$

Let us see what happens with $T_2(\varpi_2)$. Taking the $k+1$-derivatives,

$$\begin{aligned}
\partial_\alpha^{k+1} T_2(\varpi_2)(\alpha) &= \sum_{j=0}^{k+1} \frac{C_k}{\pi} \int_{\mathbb{R}} \partial_\alpha^{k+1-j} (\frac{\Delta z h^\perp \cdot \partial_\alpha z(\alpha)}{|\Delta z h|^2}) \partial_\alpha^j \varpi_2(\alpha-\beta) d\beta \\
&= T_2(\partial_\alpha^{k+1}\varpi_2)(\alpha) + \frac{1}{\pi}\int_{\mathbb{R}} \partial_\alpha^{k+1-j}(\frac{\Delta z h^\perp \cdot \partial_\alpha z(\alpha)}{|\Delta z h|^2})\varpi_2(\alpha-\beta)d\beta \\
&+ \sum_{j=1}^{k} \frac{C_k}{\pi}\int_{\mathbb{R}} \partial_\alpha^{k+1-j}(\frac{\Delta z h^\perp \cdot \partial_\alpha z(\alpha)}{|\Delta z h|^2})\partial_\alpha^j \varpi_2(\alpha-\beta)d\beta \\
&= T_2(\partial_\alpha^{k+1}\varpi_2)(\alpha) + \frac{1}{\pi}\int_{\mathbb{R}} \frac{\Delta z h^\perp \cdot \partial_\alpha^{k+2} z(\alpha)}{|\Delta z h|^2}\varpi_2(\alpha-\beta)d\beta \\
&+ \text{“other terms”} \\
&= T_2(\partial_\alpha^{k+1}\varpi_2)(\alpha) + J_1 + \text{“other terms”}
\end{aligned}$$

The estimate for “other terms” is straighforward. For $T_2(\partial_\alpha^{k+1}\varpi_2)(\alpha)$ we integrate by parts:

$$\begin{aligned}
T_2(\partial_\alpha^{k+1}\varpi_2)(\alpha) &= \frac{-1}{\pi}\int_{\mathbb{R}} \frac{\Delta z h^\perp \cdot \partial_\alpha z(\alpha)}{|\Delta z h|^2}\partial_\beta(\partial_\alpha^k \varpi_2(\alpha-\beta))d\beta \\
&= \frac{1}{\pi}\int_{\mathbb{R}} \partial_\beta(\frac{\Delta z h^\perp \cdot \partial_\alpha z(\alpha)}{|\Delta z h|^2})\partial_\alpha^k\varpi_2(\alpha-\beta)d\beta \\
&= -\frac{1}{\pi}\int_{\mathbb{R}} \frac{\partial_\alpha^\perp h(\alpha-\beta)\cdot\partial_\alpha z(\alpha)}{|\Delta z h|^2}\partial_\alpha^k\varpi_2(\alpha-\beta)d\beta \\
&+ \frac{2}{\pi}\int_{\mathbb{R}} \frac{\Delta z h^\perp \cdot \partial_\alpha z(\alpha)\Delta z h\cdot\partial_\alpha h(\alpha-\beta)}{|\Delta z h|^4}\partial_\alpha^k\varpi_2(\alpha-\beta)d\beta \equiv I_1 + I_2.
\end{aligned}$$

It is easy estimate I_1

$$|I_1| \le C\|\partial_\alpha z\|_{L^\infty}\|d(z,h)\|_{L^\infty}\|h\|_{H^1}\|\varpi_2\|_{H^k}.$$

For I_2, using the Cauchy inequality

$$\begin{aligned}
|I_2| &\le \frac{1}{2\pi}\int_{\mathbb{R}} \frac{(|\Delta z h|^2 + |\partial_\alpha z(\alpha)|^2)(|\Delta z h|^2 + |\partial_\alpha h(\alpha-\beta)|^2)}{|\Delta z h|^4}\partial_\alpha^k\varpi_2(\alpha-\beta)d\beta \\
&\le C\|\varpi_2\|_{H^k} + C\|d(z,h)\|_{L^\infty}\|z\|_{\mathcal{C}^1}^2\|\varpi_2\|_{H^k} + C\|d(z,h)\|_{L^\infty}\|h\|_{\mathcal{C}^1}^2\|\varpi_2\|_{H^k} + C\|d(z,h)\|_{L^\infty}^2\|z\|_{\mathcal{C}^1}^2\|h\|_{\mathcal{C}^1}^2\|\varpi_2\|_{H^k}.
\end{aligned}$$

If we use the same procedure to estimate J_1,

$$J_1 \le \frac{1}{2\pi}\int_{\mathbb{R}} \frac{|\Delta z h|^2 + |\partial_\alpha^{k+2} z(\alpha)|^2}{|\Delta z h|^2}\varpi_2(\alpha-\beta)d\beta \le C\|\varpi_2\|_{L^2} + C\|d(z,h)\|_{L^\infty}\|z\|_{H^{k+2}}^2\|\varpi_2\|_{L^2}.$$

Then,

$$\|T_2(\varpi_2)\|_{H^{k+1}} \le C|||z,h|||^2\|z\|_{H^{k+2}}^2\|\varpi_2\|_{H^k}.$$

Since $T_3(\varpi_1)$ is $T_2(\varpi_2)$ changing z for h and viceverse, the estimations will be

$$\|T_3(\varpi_1)\|_{H^{k+1}} \le C|||z,h|||^2\|h\|_{H^{k+2}}^2\|\varpi_1\|_{H^k}.$$

Therefore,

$$\|\mathcal{T}\varpi\|_{H^{k+1}} \le C|||z,h|||^2(\|z\|_{H^{k+2}}^2 + \|h\|_{H^{k+2}}^2)\|\varpi\|_{H^k}.$$

Since h is fixed on time, we get the desired estimate. ■

2.3 Estimates on $BR(\varpi_1,z)_z+BR(\varpi_2,h)_z+BR(\varpi_1,z)_h+BR(\varpi_2,h)_h$

This section is devoted to show that the Birkhoff-Rott integral is as regular as $\partial_\alpha z$.

Lemma 2.3.1. *The following estimate holds*

$$\begin{aligned}&\|BR(\varpi_1,z)_z\|_{H^k}+\|BR(\varpi_2,h)_z\|_{H^k}+\|BR(\varpi_1,z)_h\|_{H^k}+\|BR(\varpi_2,h)_h\|_{H^k}\\&\leq \exp C(\|\mathcal{F}(z)\|^2_{L^\infty}+\|d(z,h)\|^2_{L^\infty}+\|z\|^2_{H^{k+1}})\end{aligned}\tag{2.20}$$

for $k\geq 2$.

Proof. The lemma 6.1 on [11] gives us,

$$\|BR(\varpi_1,z)_z\|_{H^k}\leq \exp C(\|\mathcal{F}(z)\|^2_{L^\infty}+\|z\|^2_{H^{k+1}})$$

and

$$\|BR(\varpi_2,h)_h\|_{H^k}\leq \exp C(\|\mathcal{F}(h)\|^2_{L^\infty}+\|h\|^2_{H^{k+1}}).$$

Using that $\|h\|^2_{H^{k+1}}$ and $\|\mathcal{F}(h)\|^2_{L^\infty}$ are not dependent of time,

$$\begin{aligned}&\|BR(\varpi_1,z)_z\|_{H^k}+\|BR(\varpi_2,h)_h\|_{H^k}\\&\leq \exp C(\|\mathcal{F}(z)\|^2_{L^\infty}+\|d(z,h)\|^2_{L^\infty}+\|z\|^2_{H^{k+1}}).\end{aligned}$$

Let us see what happens with $BR(\varpi_1,z)_h$ and $BR(\varpi_2,h)_z$. It is enough study one of then. For example let study $BR(\varpi_2,h)_z$. For $k=2$,

$$\begin{aligned}\|BR(\varpi_2,h)_z\|_{L^2}&\leq C\|d(z,h)\|_{L^\infty}(\|z\|_{L^2}+\|h\|_{L^2})\|\varpi_2\|_{L^2}\\&\leq \exp C(\|\mathcal{F}(z)\|^2_{L^\infty}+\|d(z,h)\|^2_{L^\infty}+\|z\|^2_{H^1}).\end{aligned}$$

If we take two derivatives, we get $BR(\varpi_2,h)_z=B_1+B_2+B_3+$ "other terms" where

$$\begin{aligned}B_1&=\frac{1}{2\pi}\int_{\mathbb{R}}\frac{(\Delta zh)^\perp}{|\Delta zh|^2}\partial^2_\alpha\varpi_2(\alpha-\beta)d\beta,\\B_2&=\frac{1}{2\pi}\int_{\mathbb{R}}\frac{(\partial^2_\alpha z(\alpha)-\partial^2_\alpha h(\alpha-\beta))^\perp}{|\Delta zh|^2}\varpi_2(\alpha-\beta)d\beta,\\B_3&=-\frac{1}{\pi}\int_{\mathbb{R}}\frac{(\Delta zh)^\perp(\Delta zh\cdot(\partial^2_\alpha z(\alpha)-\partial^2_\alpha h(\alpha-\beta)))}{|\Delta zh|^4}\varpi_2(\alpha-\beta)d\beta.\end{aligned}$$

Using the estimations in ϖ and the distance of z and h,

$$\begin{aligned}&\|B_1\|_{L^2}\leq C\|d(z,h)\|^{\frac{1}{2}}_{L^\infty}\|\varpi_2\|_{H^2}\\&\leq \exp C(\|\mathcal{F}(z)\|^2_{L^\infty}+\|d(z,h)\|^2_{L^\infty}+\|z\|^2_{H^3}).\end{aligned}$$

For B_2,

$$\begin{aligned}\|B_2\|_{L^2} &\leq C\|d(z,h)\|_{L^\infty}(\|z\|_{H^2}+\|h\|_{H^2})\|\varpi_2\|_{L^2}\\ &\leq \exp C(\|\mathcal{F}(z)\|^2_{L^\infty}+\|d(z,h)\|^2_{L^\infty}+\|z\|^2_{H^2}).\end{aligned}$$

And B_3 will be the same,

$$\begin{aligned}\|B_3\|_{L^2} &\leq C\|d(z,h)\|_{L^\infty}(\|z\|_{H^2}+\|h\|_{H^2})\|\varpi_2\|_{L^2}\\ &\leq \exp C(\|\mathcal{F}(z)\|^2_{L^\infty}+\|d(z,h)\|^2_{L^\infty}+\|z\|^2_{H^2}).\end{aligned}$$

These estimations allow us to get the desire result. ■

2.4 Estimates for the L^2 norm of the curve

We have

$$z_t(\alpha)=BR(\varpi_1,z)_z+BR(\varpi_2,h)_z+c(\alpha)\partial_\alpha z(\alpha)$$

where

$$\begin{aligned}c(\alpha)=&\frac{\alpha+\pi}{2\pi A(t)}\int_{\mathbb{T}}\partial_\alpha z(\beta)\cdot(\partial_\alpha BR(\varpi_1,z)_z+\partial_\alpha BR(\varpi_2,h)_z)d\beta\\ &-\int_\pi^\alpha\frac{\partial_\alpha z(\beta)}{A(t)}\cdot(\partial_\alpha BR(\varpi_1,z)_z+\partial_\alpha BR(\varpi_2,h)_z)d\beta.\end{aligned}$$

Recall that $A(t)=|\partial_\alpha z(\alpha)|^2$. Then,

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}}|z(\alpha)|^2d\alpha=\int_{\mathbb{T}}z(\alpha)\cdot z_t(\alpha)d\alpha=\int_{\mathbb{T}}z(\alpha)\cdot BR(\varpi_1,z)_zd\alpha\\ &+\int_{\mathbb{T}}z(\alpha)\cdot BR(\varpi_2,h)_zd\alpha+\int_{\mathbb{T}}c(\alpha)z(\alpha)\cdot\partial_\alpha z(\alpha)d\alpha\equiv I_1+I_2+I_3.\end{aligned}$$

Taking $I_1+I_2\leq\|z\|_{L^2}(\|BR(\varpi_1,z)_z\|_{L^2}+\|BR(\varpi_2,h)_z\|_{L^2})$ and the inequality (2.20) allow us to write,

$$I_1+I_2\leq\exp C(\|\mathcal{F}(z)\|^2_{L^\infty}+\|d(z,h)\|^2_{L^\infty}+\|z\|^2_{H^1}).$$

Next we get,

$$\begin{aligned}I_3&\leq A^{\frac{1}{2}}(t)\|c\|_{L^\infty}\int_{\mathbb{T}}|z(\alpha)|d\alpha\leq 2\int_{\mathbb{T}}|BR(\varpi_1,z)_z|+|BR(\varpi_2,h)_z|d\alpha\int_{\mathbb{T}}|z(\alpha)|d\alpha\\ &\leq\exp C(\|\mathcal{F}(z)\|^2_{L^\infty}+\|d(z,h)\|^2_{L^\infty}+\|z\|^2_{H^1}).\end{aligned}$$

Therefore,

$$\frac{d}{dt}\|z\|^2_{L^2}(t)\leq\exp C(\|\mathcal{F}(z)\|^2_{L^\infty}+\|d(z,h)\|^2_{L^\infty}+\|z\|^2_{H^3}).$$

2.5 Estimates on the H^3 norm

Taking the 3 derivatives on the curve, we get

$$\int_{\mathbb{T}}\partial^3_\alpha z(\alpha)\cdot\partial^3_\alpha z_t(\alpha)d\alpha=\int_{\mathbb{T}}\partial^3_\alpha z(\alpha)\cdot\partial^3_\alpha BR(\varpi_1,z)_zd\alpha$$

$$
\begin{aligned}
&+\int_{\mathbb{T}}\partial_\alpha^3 z(\alpha)\cdot\partial_\alpha^3 BR(\varpi_2,h)_z d\alpha+\int_{\mathbb{T}}\partial_\alpha^3 z(\alpha)\cdot\partial_\alpha^3(c(\alpha)\partial_\alpha z(\alpha))d\alpha\\
&\equiv I_1+I_2+I_3.
\end{aligned}
$$

Here and in the next section we will study I_1+I_2. We shall estimate I_3 in section 2.5.2. Let estimate first the term I_2. We can split $I_2=J_1+J_2+J_3+J_4$, where

$$
\begin{aligned}
J_1&=\frac{1}{2\pi}\int_{\mathbb{T}}\int_{\mathbb{R}}\partial_\alpha^3 z(\alpha)\cdot\partial_\alpha^3\Big(\frac{(z(\alpha)-h(\alpha-\beta))^\perp}{|z(\alpha)-h(\alpha-\beta)|^2}\Big)\varpi_2(\alpha-\beta)d\beta d\alpha,\\
J_2&=\frac{3}{2\pi}\int_{\mathbb{T}}\int_{\mathbb{R}}\partial_\alpha^3 z(\alpha)\cdot\partial_\alpha^2\Big(\frac{(z(\alpha)-h(\alpha-\beta))^\perp}{|z(\alpha)-h(\alpha-\beta)|^2}\Big)\partial_\alpha\varpi_2(\alpha-\beta)d\beta d\alpha,\\
J_3&=\frac{3}{2\pi}\int_{\mathbb{T}}\int_{\mathbb{R}}\partial_\alpha^3 z(\alpha)\cdot\partial_\alpha\Big(\frac{(z(\alpha)-h(\alpha-\beta))^\perp}{|z(\alpha)-h(\alpha-\beta)|^2}\Big)\partial_\alpha^2\varpi_2(\alpha-\beta)d\beta d\alpha,\\
J_4&=\frac{1}{2\pi}\int_{\mathbb{T}}\int_{\mathbb{R}}\partial_\alpha^3 z(\alpha)\cdot\frac{(z(\alpha)-h(\alpha-\beta))^\perp}{|z(\alpha)-h(\alpha-\beta)|^2}\partial_\alpha^3\varpi_2(\alpha-\beta)d\beta d\alpha.
\end{aligned}
$$

The most singular terms for J_1 are:

$$
\begin{aligned}
J_1^1&=\frac{1}{4\pi}\int_{\mathbb{T}}\int_{\mathbb{R}}\partial_\alpha^3 z(\alpha)\cdot\Big(\frac{(\partial_\alpha^3 z(\alpha)-\partial_\alpha^3 h(\alpha-\beta))^\perp}{|z(\alpha)-h(\alpha-\beta)|^2}\Big)\varpi_2(\alpha-\beta)d\beta d\alpha,\\
J_1^2&=-\frac{1}{2\pi}\int_{\mathbb{T}}\int_{\mathbb{R}}\partial_\alpha^3 z(\alpha)\cdot\Big(\frac{(\Delta zh)^\perp\Delta zh\cdot(\partial_\alpha^3 z(\alpha)-\partial_\alpha^3 h(\alpha-\beta))}{|z(\alpha)-h(\alpha-\beta)|^4}\Big)\varpi_2(\alpha-\beta)d\beta d\alpha.
\end{aligned}
$$

Using $\partial_\alpha^3 z\cdot\partial_\alpha^3 z^\perp=0$,

$$|J_1^1|\le C\|d(z,h)\|_{L^\infty}\|z\|_{H^3}\|h\|_{H^3}\|\varpi_2\|_{L^\infty}.$$

Using the same technique,

$$|J_1^2|\le C\|d(z,h)\|_{L^\infty}\|z\|_{H^3}(\|z\|_{H^3}+\|h\|_{H^3})\|\varpi_2\|_{L^\infty}.$$

Then,

$$J_1\le\exp C(\|\mathcal{F}(z)\|^2_{L^\infty}+\|d(z,h)\|^2_{L^\infty}+\|z\|^2_{H^3}).$$

The most singular term in J_2 is:

$$
\begin{aligned}
J_2^1&=C\int_{\mathbb{T}}\int_{\mathbb{R}}\partial_\alpha^3 z(\alpha)\cdot\frac{(\partial_\alpha^2 z(\alpha)-\partial_\alpha^2 h(\alpha-\beta))^\perp}{|\Delta zh|^2}\partial_\alpha\varpi_2(\alpha-\beta)d\beta d\alpha\\
&\le C\|d(z,h)\|_{L^\infty}\|z\|_{H^3}(\|z\|_{\mathcal{C}^2}+\|h\|_{\mathcal{C}^2})\|\varpi_2\|_{H^1}.
\end{aligned}
$$

Then,

$$J_2\le\exp C(\|\mathcal{F}(z)\|^2_{L^\infty}+\|d(z,h)\|^2_{L^\infty}+\|z\|^2_{H^3}).$$

For J_3,

$$J_3^1=C\int_{\mathbb{T}}\int_{\mathbb{R}}\partial_\alpha^3 z(\alpha)\cdot\frac{(\partial_\alpha z(\alpha)-\partial_\alpha h(\alpha-\beta))^\perp}{|\Delta zh|^2}\partial_\alpha^2\varpi_2(\alpha-\beta)d\beta d\alpha$$

$$
\begin{aligned}
&\leq C\|d(z,h)\|_{L^\infty}(\|z\|_{\mathcal{C}^1}+\|h\|_{\mathcal{C}^1})\|z\|_{H^3}\|\varpi_2\|_{H^2}\\
&\leq \exp C(\|\mathcal{F}(z)\|^2_{L^\infty}+\|d(z,h)\|^2_{L^\infty}+\|z\|^2_{H^3}).
\end{aligned}
$$

Using integration by parts we will estimate J_4,

$$
\begin{aligned}
J_4 &= -\frac{1}{2\pi}\int_{\mathbb{T}}\int_{\mathbb{R}}\partial^3_\alpha z(\alpha)\cdot\frac{(\Delta zh)^\perp}{|\Delta zh|^2}\partial_\beta\partial^2_\alpha\varpi_2(\alpha-\beta)d\beta d\alpha\\
&= -\frac{1}{2\pi}\int_{\mathbb{T}}\int_{\mathbb{R}}\partial^3_\alpha z(\alpha)\cdot\frac{\partial^\perp_\alpha h(\alpha-\beta)}{|\Delta zh|^2}\partial^2_\alpha\varpi_2(\alpha-\beta)d\beta d\alpha\\
&+\frac{1}{\pi}\int_{\mathbb{T}}\int_{\mathbb{R}}\partial^3_\alpha z(\alpha)\cdot\frac{(\Delta zh)^\perp\Delta zh\cdot\partial_\alpha h(\alpha-\beta)}{|\Delta zh|^4}\partial_\beta\partial^2_\alpha\varpi_2(\alpha-\beta)d\beta d\alpha\equiv J^1_4+J^2_4.
\end{aligned}
$$

It is clear,

$$
\begin{aligned}
J^1_4 &\leq C\|d(z,h)\|_{L^\infty}\|h\|_{\mathcal{C}^1}\|z\|_{H^3}\|\varpi_2\|_{H^2},\\
J^2_4 &\leq C\|d(z,h)\|_{L^\infty}\|h\|_{\mathcal{C}^1}\|z\|_{H^3}\|\varpi_2\|_{H^2}.
\end{aligned}
$$

Therefore,

$$
I_2\leq \exp C(\|\mathcal{F}(z)\|^2_{L^\infty}+\|d(z,h)\|^2_{L^\infty}+\|z\|^2_{H^3}).
$$

2.5.1 Estimations on I_1

In these estimations we will found the Rayleigh-Taylor condition. We can split I_1 in the following terms:

$$
\begin{aligned}
I^1_1 &= \frac{1}{2\pi}\int_{\mathbb{T}}\int_{\mathbb{R}}\partial^3_\alpha z(\alpha)\cdot\partial^3_\alpha\Big(\frac{(z(\alpha)-z(\alpha-\beta))^\perp}{|z(\alpha)-z(\alpha-\beta)|^2}\Big)\varpi_1(\alpha-\beta)d\beta d\alpha,\\
I^2_1 &= \frac{3}{2\pi}\int_{\mathbb{T}}\int_{\mathbb{R}}\partial^3_\alpha z(\alpha)\cdot\partial^2_\alpha\Big(\frac{(z(\alpha)-z(\alpha-\beta))^\perp}{|z(\alpha)-z(\alpha-\beta)|^2}\Big)\partial_\alpha\varpi_1(\alpha-\beta)d\beta d\alpha,\\
I^3_1 &= \frac{3}{2\pi}\int_{\mathbb{T}}\int_{\mathbb{R}}\partial^3_\alpha z(\alpha)\cdot\partial_\alpha\Big(\frac{(z(\alpha)-z(\alpha-\beta))^\perp}{|z(\alpha)-z(\alpha-\beta)|^2}\Big)\partial^2_\alpha\varpi_1(\alpha-\beta)d\beta d\alpha,\\
I^4_1 &= \frac{1}{2\pi}\int_{\mathbb{T}}\int_{\mathbb{R}}\partial^3_\alpha z(\alpha)\cdot\frac{(z(\alpha)-z(\alpha-\beta))^\perp}{|z(\alpha)-z(\alpha-\beta)|^2}\partial^3_\alpha\varpi_1(\alpha-\beta)d\beta d\alpha.
\end{aligned}
$$

The terms I^1_1, I^2_1 and I^3_1 can be estimated like in the section 7.2 in [11]. Then we have to estimate I^4_1.

$$
\begin{aligned}
I^4_1 &= \frac{1}{2\pi}\int_{\mathbb{T}}\int_{\mathbb{R}}\partial^3_\alpha z(\alpha)\cdot\Big(\frac{(\Delta z)^\perp}{|\Delta z|^2}-\frac{\partial^\perp_\alpha z(\alpha)}{\beta|\partial_\alpha z(\alpha)|^2}\Big)\partial^3_\alpha\varpi_1(\alpha-\beta)d\beta d\alpha\\
&+\frac{1}{2\pi}\int_{\mathbb{T}}\int_{\mathbb{R}}\partial^3_\alpha z(\alpha)\cdot\Big(\frac{\partial^\perp_\alpha z(\alpha)}{\beta|\partial_\alpha z(\alpha)|^2}\Big)\partial^3_\alpha\varpi_1(\alpha-\beta)d\beta d\alpha\equiv I^{41}_1+I^{42}_1.
\end{aligned}
$$

Using integration by parts,

$$I_1^{41} = -\frac{1}{2\pi}\int_{\mathbb{T}}\int_{\mathbb{R}} \partial_\alpha^3 z(\alpha)\cdot(\frac{(\Delta z)^\perp}{|\Delta z|^2} - \frac{\partial_\alpha^\perp z(\alpha)}{\beta|\partial_\alpha z(\alpha)|^2})\partial_\beta\partial_\alpha^2\varpi_1(\alpha-\beta)d\beta d\alpha$$
$$= \frac{1}{2\pi}\int_{\mathbb{T}}\int_{\mathbb{R}} \partial_\alpha^3 z(\alpha)\cdot\partial_\beta(\frac{(\Delta z)^\perp}{|\Delta z|^2} - \frac{\partial_\alpha^\perp z(\alpha)}{\beta|\partial_\alpha z(\alpha)|^2})\partial_\alpha^2\varpi_1(\alpha-\beta)d\beta d\alpha.$$

If we decompose,

$$\partial_\beta(\frac{(\Delta z)^\perp}{|\Delta z|^2} - \frac{\partial_\alpha^\perp z(\alpha)}{|\partial_\alpha z(\alpha)|^2\beta})$$
$$= \frac{(\Delta\partial_\alpha z)^\perp}{|\Delta z|^2} + \partial_\alpha^\perp z(\alpha)(\frac{1}{|\Delta z|^2} - \frac{1}{|\partial_\alpha z(\alpha)|^2\beta^2}) - 2\frac{(\Delta z)^\perp\Delta z\cdot\Delta\partial_\alpha z}{|\Delta z|^4}$$
$$- 2\frac{(\Delta z)^\perp(\Delta z - \beta\partial_\alpha z(\alpha))\cdot\partial_\alpha z(\alpha)}{|\Delta z|^4} - 2\frac{(\Delta z - \beta\partial_\alpha z(\alpha))^\perp\beta|\partial_\alpha z(\alpha)|^2}{|\Delta z|^4}$$
$$+ (\frac{2\partial_\alpha^\perp z(\alpha)}{|\partial_\alpha z(\alpha)|^2\beta^2} - \frac{2\beta^2\partial_\alpha^\perp z(\alpha)|\partial_\alpha z(\alpha)|^2}{|\Delta z|^4})$$
$$\equiv F_1(\alpha,\beta) + F_2(\alpha,\beta) + F_3(\alpha,\beta) + F_4(\alpha,\beta) + F_5(\alpha,\beta) + F_6(\alpha,\beta).$$

We have

$$I_1^{41} = \frac{1}{2\pi}\int_{\mathbb{T}}\int_{\mathbb{R}} \partial_\alpha^3 z(\alpha)\cdot F_1(\alpha,\beta)\partial_\alpha^2\varpi_1(\alpha-\beta)d\beta d\alpha$$
$$+ \frac{1}{2\pi}\int_{\mathbb{T}}\int_{\mathbb{R}} \partial_\alpha^3 z(\alpha)\cdot F_2(\alpha,\beta)\partial_\alpha^2\varpi_1(\alpha-\beta)d\beta d\alpha$$
$$+ \frac{1}{2\pi}\int_{\mathbb{T}}\int_{\mathbb{R}} \partial_\alpha^3 z(\alpha)\cdot F_3(\alpha,\beta)\partial_\alpha^2\varpi_1(\alpha-\beta)d\beta d\alpha$$
$$+ \frac{1}{2\pi}\int_{\mathbb{T}}\int_{\mathbb{R}} \partial_\alpha^3 z(\alpha)\cdot F_4(\alpha,\beta)\partial_\alpha^2\varpi_1(\alpha-\beta)d\beta d\alpha$$
$$+ \frac{1}{2\pi}\int_{\mathbb{T}}\int_{\mathbb{R}} \partial_\alpha^3 z(\alpha)\cdot F_5(\alpha,\beta)\partial_\alpha^2\varpi_1(\alpha-\beta)d\beta d\alpha$$
$$+ \frac{1}{2\pi}\int_{\mathbb{T}}\int_{\mathbb{R}} \partial_\alpha^3 z(\alpha)\cdot F_6(\alpha,\beta)\partial_\alpha^2\varpi_1(\alpha-\beta)d\beta d\alpha$$
$$\equiv I_1^{411} + I_1^{412} + I_1^{413} + I_1^{414} + I_1^{415} + I_1^{416}.$$

Computing

$$F_1(\alpha,\beta) - \frac{\partial_\alpha^2 z(\alpha)^\perp}{\beta|\partial_\alpha z(\alpha)|^2} = \frac{\beta^2\int_0^1 \partial_\alpha^2 z(\alpha-\beta ts)^\perp - \partial_\alpha^2 z(\alpha)^\perp dsdt}{|\Delta z|^2}$$
$$+ \frac{\beta^2\partial_\alpha^2 z(\alpha)^\perp\int_0^1\int_0^1(1-t)\partial_\alpha^2 z(\psi)dsdt\cdot\int_0^1\partial_\alpha z(\alpha) + \partial_\alpha z(\alpha-\beta+\beta t)dt}{|\Delta z|^2|\partial_\alpha z(\alpha)|^2}$$

where $\psi = \alpha - \beta + \beta t + s\beta + \beta ts$, we get

$$I_1^{411} \leq C\|\mathcal{F}(z)\|_{L^\infty}\|z\|_{\mathcal{C}^{2,\delta}}\|z\|_{H^3}\|\varpi_1\|_{H^2} + C\|\mathcal{F}(z)\|_{L^\infty}^{\frac{3}{2}}\|z\|_{\mathcal{C}^2}\|z\|_{H^3}\|\varpi_1\|_{H^2}$$
$$+ \frac{1}{2\pi}\int_{\mathbb{T}} \partial_\alpha^3 z(\alpha)\cdot\frac{\partial_\alpha^2 z(\alpha)^\perp}{|\partial_\alpha z(\alpha)|^2}H(\partial_\alpha^2\varpi_1)(\alpha)d\alpha.$$

Then,

$$I_1^{411} \leq \exp C(\|\mathcal{F}(z)\|^2_{L^\infty} + \|d(z,h)\|^2_{L^\infty} + \|z\|^2_{H^3}).$$

Analogously,

$$I_1^{412} \leq C\|\mathcal{F}(z)\|^{\frac{3}{2}}_{L^\infty}\|z\|^2_{\mathcal{C}^2}\|z\|_{H^3}\|\varpi_1\|_{H^2} + \frac{1}{\pi}\int_{\mathbb{T}} \partial^3_\alpha z(\alpha)\cdot\partial^\perp_\alpha z(\alpha)\frac{\partial^2_\alpha z(\alpha)\cdot\partial_\alpha z(\alpha)}{|\partial_\alpha z(\alpha)|^2}H(\partial^2_\alpha\varpi_1)(\alpha)d\alpha.$$

Using the fact that we can split F_3, with $\phi = \alpha-\beta+\beta t$ and $\psi = \alpha-\beta+\beta t+s\beta-\beta ts$.

$$\begin{aligned} F_3(\alpha,\beta) &= -2\beta^3\int_0^1 \partial^\perp_\alpha z(\phi)dt\int_0^1 \partial_\alpha z(\phi)dt\cdot\int_0^1 \partial^2_\alpha z(\phi)dt\Big(\frac{1}{|\Delta z|^4}-\frac{1}{\beta^4|\partial_\alpha z(\alpha)|^4}\Big)\\ &-\frac{2\int_0^1\partial^\perp_\alpha z(\phi)dt\int_0^1\partial_\alpha z(\phi)dt\cdot\int_0^1\partial^2_\alpha z(\phi)dt}{\beta|\partial_\alpha z(\alpha)|^4}\end{aligned}$$

and

$$\begin{aligned}&\frac{1}{|\Delta z|^4}-\frac{1}{\beta^4|\partial_\alpha z(\alpha)|^4}\\ &=\frac{\beta\int_0^1\int_0^1\partial^2_\alpha z(\psi)(t-1)dtds\cdot\int_0^1\partial_\alpha z(\alpha)+\partial_\alpha z(\phi)dt\int_0^1|\partial_\alpha z(\alpha)|^2+|\partial_\alpha z(\phi)|^2}{|\Delta z|^4|\partial_\alpha z(\alpha)|^4},\end{aligned}$$

we get,

$$\begin{aligned} I_1^{413} &\leq \exp C(\|\mathcal{F}(z)\|^2_{L^\infty}+\|d(z,h)\|^2_{L^\infty}+\|z\|^2_{H^3})\\ &-\frac{1}{\pi}\int_{\mathbb{T}}\partial^3_\alpha z(\alpha)\cdot\partial^\perp_\alpha z(\alpha)\frac{\partial_\alpha z(\alpha)\cdot\partial^2_\alpha z(\alpha)}{|\partial_\alpha z(\alpha)|^4}H(\partial^2_\alpha\varpi_1)(\alpha)d\alpha\\ &\leq \exp C(\|\mathcal{F}(z)\|^2_{L^\infty}+\|d(z,h)\|^2_{L^\infty}+\|z\|^2_{H^3})+C\|\mathcal{F}(z)\|_{L^\infty}\|z\|_{\mathcal{C}^2}\|z\|_{H^3}\|\varpi_1\|_{H^2}.\end{aligned}$$

For I_1^{414} and I_1^{415} it is easy to see that it is bounded in the same way as the above terms. Let us study the term I_1^{416}. Since,

$$\begin{aligned} -\frac{1}{2}F_6(\alpha,\beta) &= \partial^\perp_\alpha z(\alpha)\frac{\beta^4|\partial_\alpha z(\alpha)|^4-|\Delta z|^4}{|\Delta z|^4|\partial_\alpha z(\alpha)|^2\beta^2}\\ &=\partial^\perp_\alpha z(\alpha)\frac{\beta^3\int_0^1\int_0^1(\partial^2_\alpha z(\psi)-\partial^2_\alpha z(\alpha))(1-s)dtds\int_0^1[\partial_\alpha z(\alpha)+\partial_\alpha z(\phi)]ds\int_0^1[|\partial_\alpha z(\alpha)|^2+|\partial_\alpha z(\phi)|^2]ds}{|\Delta z|^4|\partial_\alpha z(\alpha)|^2}\\ &+\frac{\partial^\perp_\alpha z(\alpha)}{2}\frac{\beta^4\partial^2_\alpha z(\alpha)\int_0^1\int_0^1\partial^2_\alpha z(\eta)(s-1)dtds\int_0^1[|\partial_\alpha z(\alpha)|^2+|\partial_\alpha z(\phi)|^2]ds}{|\Delta z|^4|\partial_\alpha z(\alpha)|^2}\\ &+\partial^\perp_\alpha z(\alpha)\frac{\beta^4\partial^2_\alpha z(\alpha)\partial_\alpha z(\alpha)\int_0^1\int_0^1\partial_\alpha z(\eta)\cdot\partial^2_\alpha z(\eta)(s-1)dtds}{|\Delta z|^4|\partial_\alpha z(\alpha)|^2}+\partial^\perp_\alpha z(\alpha)\frac{\beta^3\partial^2_\alpha z(\alpha)\partial_\alpha z(\alpha)}{|\Delta z|^4}\\ &\equiv U_1(\alpha,\beta)+U_2(\alpha,\beta)+U_3(\alpha,\beta)+U_4(\alpha,\beta)\end{aligned}$$

we get,

$$\begin{aligned} I_1^{416} &= -\frac{1}{\pi}\int_{\mathbb{T}}\int_{\mathbb{R}}\partial^3_\alpha z(\alpha)\cdot U_1(\alpha,\beta)\partial^2_\alpha\varpi(\alpha-\beta)d\alpha d\beta-\frac{1}{\pi}\int_{\mathbb{T}}\int_{\mathbb{R}}\partial^3_\alpha z(\alpha)\cdot U_2(\alpha,\beta)\partial^2_\alpha\varpi(\alpha-\beta)d\alpha d\beta\\ &-\frac{1}{\pi}\int_{\mathbb{T}}\int_{\mathbb{R}}\partial^3_\alpha z(\alpha)\cdot U_3(\alpha,\beta)\partial^2_\alpha\varpi(\alpha-\beta)d\alpha d\beta-\frac{1}{\pi}\int_{\mathbb{T}}\int_{\mathbb{R}}\partial^3_\alpha z(\alpha)\cdot U_4(\alpha,\beta)\partial^2_\alpha\varpi(\alpha-\beta)d\alpha d\beta\\ &\equiv Q_1+Q_2+Q_3+Q_4.\end{aligned}$$

It is clear,

$$
\begin{aligned}
&Q_1 \leq C\|\mathcal{F}(z)\|_{L^\infty}\|z\|_{\mathcal{C}^{2,\delta}}\|z\|_{H^3}\|\varpi_1\|_{H^2},\\
&Q_2 \leq C\|\mathcal{F}(z)\|^2_{L^\infty}\|z\|^2_{\mathcal{C}^2}\|z\|_{\mathcal{C}^1}\|z\|_{H^3}\|\varpi_1\|_{H^2},\\
&Q_3 \leq C\|\mathcal{F}(z)\|^2_{L^\infty}\|z\|^2_{\mathcal{C}^2}\|z\|_{\mathcal{C}^1}\|z\|_{H^3}\|\varpi_1\|_{H^2},\\
&Q_4 \leq C\|\mathcal{F}(z)\|^2_{L^\infty}\|z\|^2_{\mathcal{C}^2}\|z\|_{\mathcal{C}^1}\|z\|_{H^3}\|\varpi_1\|_{H^2}\\
&-\frac{1}{\pi}\int_{\mathbb{T}}\partial^3_\alpha z(\alpha)\cdot\partial^\perp_\alpha z(\alpha)\frac{\partial^2_\alpha z(\alpha)\cdot\partial_\alpha z(\alpha)}{|\partial_\alpha z(\alpha)|^4}H(\partial^2_\alpha\varpi_1)(\alpha)d\alpha.
\end{aligned}
$$

Therefore,

$$I_1^{41} \leq \exp C(\|\mathcal{F}(z)\|^2_{L^\infty} + \|d(z,h)\|^2_{L^\infty} + \|z\|^2_{H^3}).$$

Now, we have to study I_1^{42}. Using $\partial_\alpha H = \Lambda$,

$$
\begin{aligned}
I_1^{42} &= \frac{1}{2\pi}\int_{\mathbb{T}}\partial^3_\alpha z(\alpha)\cdot\frac{\partial^\perp_\alpha z(\alpha)}{|\partial_\alpha z(\alpha)|^2}H(\partial^3_\alpha\varpi_1)(\alpha)d\alpha = \frac{1}{2\pi}\int_{\mathbb{T}}\partial^3_\alpha z(\alpha)\cdot\frac{\partial^\perp_\alpha z(\alpha)}{|\partial_\alpha z(\alpha)|^2}\Lambda(\partial^2_\alpha\varpi_1)(\alpha)d\alpha\\
&= \frac{1}{2\pi A(t)}\int_{\mathbb{T}}\Lambda(\partial^3_\alpha z(\alpha)\cdot\partial^\perp_\alpha z(\alpha))\partial^2_\alpha\varpi_1(\alpha)d\alpha.
\end{aligned}
$$

Since $\varpi_1 = -\lambda_1 T_1(\varpi_1) - \lambda_1 T_2(\varpi_2) - N\partial_\alpha z_2(\alpha)$ we split I_1^{42},

$$
\begin{aligned}
I_1^{421} &= \frac{-N}{2\pi A(t)}\int_{\mathbb{T}}\Lambda(\partial^3_\alpha z(\alpha)\cdot\partial^\perp_\alpha z(\alpha))\partial^3_\alpha z_2(\alpha)d\alpha,\\
I_1^{422} &= \frac{-\lambda_1}{2\pi A(t)}\int_{\mathbb{T}}\Lambda(\partial^3_\alpha z\cdot\partial^\perp_\alpha z)(\alpha)(\partial^2_\alpha T_1(\varpi_1) + \partial^2_\alpha T_2(\varpi_2))d\alpha.
\end{aligned}
$$

We can write $I_1^{421} = L_1 + L_2$ where

$$
\begin{aligned}
L_1 &= \frac{N}{2\pi A(t)}\int_{\mathbb{T}}\Lambda(\partial^3_\alpha z_1\partial_\alpha z_2)(\alpha)\partial^3_\alpha z_2(\alpha)d\alpha,\\
L_2 &= \frac{-N}{2\pi A(t)}\int_{\mathbb{T}}\Lambda(\partial^3_\alpha z_2\partial_\alpha z_1)(\alpha)\partial^3_\alpha z_2(\alpha)d\alpha.
\end{aligned}
$$

Use the commutator estimation allow us,

$$
\begin{aligned}
L_1 &\leq C\|z\|_{\mathcal{C}^{2,\delta}}\|z\|^2_{H^3} + \frac{N}{2\pi A(t)}\int_{\mathbb{T}}\partial_\alpha z_2(\alpha)\partial^3_\alpha z_2(\alpha)\Lambda(\partial^3_\alpha z_1)(\alpha)d\alpha\\
&\leq \exp C(\|\mathcal{F}(z)\|^2_{L^\infty} + \|d(z,h)\|^2_{L^\infty} + \|z\|^2_{H^3}) + L_1^1.
\end{aligned}
$$

Since $A(t) = |\partial_\alpha z(\alpha)|^2$ if we derivate twice with ∂_α we get

$$\partial_\alpha z_2(\alpha)\partial^3_\alpha z_2(\alpha) = -\partial_\alpha z_1(\alpha)\partial^3_\alpha z_1(\alpha) - |\partial^2_\alpha z(\alpha)|^2.$$

Then,

$$
\begin{aligned}
L_1^1 &= -\frac{N}{2\pi A(t)}\int_{\mathbb{T}}|\partial_\alpha^2 z(\alpha)|^2\Lambda(\partial_\alpha^3 z_1)(\alpha)d\alpha - \frac{N}{2\pi A(t)}\int_{\mathbb{T}}\partial_\alpha z_1(\alpha)\partial_\alpha^3 z_1(\alpha)\Lambda(\partial_\alpha^3 z_1)(\alpha)d\alpha\\
&= \frac{N}{2\pi A(t)}\int_{\mathbb{T}}\partial_\alpha|\partial_\alpha^2 z(\alpha)|^2 H(\partial_\alpha^3 z_1)(\alpha)d\alpha - \frac{N}{2\pi A(t)}\int_{\mathbb{T}}\partial_\alpha z_1(\alpha)\partial_\alpha^3 z_1(\alpha)\Lambda(\partial_\alpha^3 z_1)(\alpha)d\alpha\\
&\leq C\|\mathcal{F}(z)\|_{L^\infty}\|z\|_{\mathcal{C}^2}\|z\|_{H^3}^2 - \frac{N}{2\pi A(t)}\int_{\mathbb{T}}\partial_\alpha z_1(\alpha)\partial_\alpha^3 z_1(\alpha)\Lambda(\partial_\alpha^3 z_1)(\alpha)d\alpha.
\end{aligned}
$$

In the same way, using the commutator estimation we have,

$$
L_2 \leq C\|\mathcal{F}(z)\|_{L^\infty}\|z\|_{\mathcal{C}^{2,\delta}}\|z\|_{H^3}^2 - \frac{N}{2\pi A(t)}\int_{\mathbb{T}}\partial_\alpha z_1(\alpha)\partial_\alpha^3 z_2(\alpha)\Lambda(\partial_\alpha^3 z_2)(\alpha)d\alpha.
$$

Therefore,

$$
\begin{aligned}
I_1^{421} &\leq \exp C(\|\mathcal{F}(z)\|_{L^\infty}^2 + \|d(z,h)\|_{L^\infty}^2 + \|z\|_{H^3}^2)\\
&- \frac{N}{2\pi A(t)}\int_{\mathbb{T}}\partial_\alpha z_1(\alpha)\partial_\alpha^3 z(\alpha)\cdot\Lambda(\partial_\alpha^3 z)(\alpha)d\alpha.
\end{aligned}
$$

Here we can observe that a part of the Rayleigh-Taylor condition appears. Let us estimate the term I_1^{422}. We can split this term in

$$
\begin{aligned}
L_3 &= \frac{-\lambda_1}{\pi A(t)}\int_{\mathbb{T}}\Lambda(\partial_\alpha^3 z\cdot\partial_\alpha^\perp z)(\alpha)(\partial_\alpha^2 BR(\varpi_1,z)_z + \partial_\alpha^2 BR(\varpi_2,h)_z)\cdot\partial_\alpha z(\alpha)d\alpha,\\
L_4 &= \frac{-\lambda_1}{\pi A(t)}\int_{\mathbb{T}}\Lambda(\partial_\alpha^3 z\cdot\partial_\alpha^\perp z)(\alpha)(\partial_\alpha BR(\varpi_1,z)_z + \partial_\alpha BR(\varpi_2,h)_z)\cdot\partial_\alpha^2 z(\alpha)d\alpha,\\
L_5 &= \frac{-\lambda_1}{\pi A(t)}\int_{\mathbb{T}}\Lambda(\partial_\alpha^3 z\cdot\partial_\alpha^\perp z)(\alpha)(BR(\varpi_1,z)_z + BR(\varpi_2,h)_z)\cdot\partial_\alpha^3 z(\alpha)d\alpha.
\end{aligned}
$$

We will estimate $L_3 + L_4$ and then we will find the rest of the R-T condition in the estimations of the term L_5. For L_3, using integration by parts,

$$
\begin{aligned}
L_3 &= \frac{\lambda_1}{\pi A(t)}\int_{\mathbb{T}}H(\partial_\alpha^3 z\cdot\partial_\alpha^\perp z)(\alpha)(\partial_\alpha^2 BR(\varpi_1,z)_z + \partial_\alpha^2 BR(\varpi_2,h)_z)\cdot\partial_\alpha^2 z(\alpha)d\alpha\\
&+ \frac{\lambda_1}{\pi A(t)}\int_{\mathbb{T}}H(\partial_\alpha^3 z\cdot\partial_\alpha^\perp z)(\alpha)(\partial_\alpha^3 BR(\varpi_1,z)_z + \partial_\alpha^3 BR(\varpi_2,h)_z)\cdot\partial_\alpha z(\alpha)d\alpha\\
&\equiv L_3^1 + L_3^2.
\end{aligned}
$$

Directly, using (2.20)

$$
\begin{aligned}
L_3^1 &\leq C\|\mathcal{F}(z)\|_{L^\infty}\|\partial_\alpha^3 z\cdot\partial_\alpha^\perp z\|_{L^2}(\|\partial_\alpha^2 BR(\varpi_1,z)_z\|_{L^2} + \|\partial_\alpha^2 BR(\varpi_2,h)_z\|_{L^2})\|z\|_{\mathcal{C}^2}\\
&\leq \exp C(\|\mathcal{F}(z)\|_{L^\infty}^2 + \|d(z,h)\|_{L^\infty}^2 + \|z\|_{H^3}^2).
\end{aligned}
$$

For L_3^2, we write

$$
L_3^2 = \frac{\lambda_1}{\pi A(t)}\int_{\mathbb{T}}H(\partial_\alpha^3 z\cdot\partial_\alpha^\perp z)(\alpha)\partial_\alpha^3 BR(\varpi_1,z)_z\cdot\partial_\alpha z(\alpha)d\alpha
$$

$$+\frac{\lambda_1}{\pi A(t)}\int_{\mathbb{T}} H(\partial_\alpha^3 z\cdot\partial_\alpha^\perp z)(\alpha)\partial_\alpha^3 BR(\varpi_2,h)_z\cdot\partial_\alpha z(\alpha)d\alpha\equiv L_3^{21}+L_3^{22}.$$

The application of the Leibniz's rule to $\partial_\alpha^3 BR(\varpi_1,z)_z$ produces terms which can be estimated with the same tools used before. The most singular terms for L_3^{21} are

$$L_3^{211}=\frac{\lambda_1}{\pi A(t)}\int_{\mathbb{T}} H(\partial_\alpha^3 z\cdot\partial_\alpha^\perp z)(\alpha)\partial_\alpha(BR(\partial_\alpha^2\varpi_1,z)_z)\cdot\partial_\alpha z(\alpha)d\alpha,$$
$$L_3^{212}=\frac{\lambda_1}{\pi A(t)}\int_{\mathbb{T}}\int_{\mathbb{R}} H(\partial_\alpha^3 z\cdot\partial_\alpha^\perp z)(\alpha)\frac{(\Delta\partial_\alpha^3 z)^\perp\cdot\partial_\alpha z(\alpha)}{|\Delta z|^2}\varpi_1(\alpha-\beta)d\beta d\alpha,$$
$$L_3^{213}=\frac{\lambda_1}{\pi A(t)}\int_{\mathbb{T}}\int_{\mathbb{R}} H(\partial_\alpha^3 z\cdot\partial_\alpha^\perp z)(\alpha)\frac{(\Delta z)^\perp\cdot\partial_\alpha z(\alpha)\Delta z\cdot\Delta\partial_\alpha^3 z}{|\Delta z|^4}\varpi_1(\alpha-\beta)d\beta d\alpha.$$

Since,

$$\partial_\alpha(BR(\partial_\alpha^2\varpi_1,z)_z)\cdot\partial_\alpha z(\alpha)=\partial_\alpha(T_1(\partial_\alpha^2\varpi_1))-BR(\partial_\alpha^2\varpi_1,z)_z\cdot\partial_\alpha^2 z(\alpha).$$

And using the estimations on $\|\mathbb{T}\|_{L^2\times L^2\to H^1\times H^1}$ and the estimations on $BR(\varpi_1,z)_z$ we get

$$L_3^{211}\leq C\|\mathcal{F}(z)\|_{L^\infty}\|\partial_\alpha^3 z\cdot\partial_\alpha^\perp z\|_{L^2}(\|T_1(\partial_\alpha^2\varpi_1)\|_{H^1}+\|BR(\partial_\alpha^2\varpi_1,z)_z\|_{L^2}\|z\|_{\mathcal{C}^2}).$$
$$\leq\exp C(\|\mathcal{F}(z)\|_{L^\infty}^2+\|d(z,h)\|_{L^\infty}^2+\|z\|_{H^3}^2)$$

For L_3^{212} we get,

$$M_1=\frac{\lambda_1}{\pi A(t)}\int_{\mathbb{T}}\int_{\mathbb{R}} H(\partial_\alpha^3 z\cdot\partial_\alpha^\perp z)(\alpha)\Delta\partial_\alpha^3 z^\perp\cdot\partial_\alpha z(\alpha)\varpi_1(\alpha-\beta)\left(\frac{1}{|\Delta z|^2}-\frac{1}{\beta^2|\partial_\alpha z(\alpha)|^2}\right)d\beta d\alpha,$$
$$M_2=\frac{\lambda_1}{\pi A(t)}\int_{\mathbb{T}}\int_{\mathbb{R}} H(\partial_\alpha^3 z\cdot\partial_\alpha^\perp z)(\alpha)\Delta\partial_\alpha^3 z^\perp\cdot\partial_\alpha z(\alpha)\frac{\varpi_1(\alpha-\beta)}{\beta^2|\partial_\alpha z(\alpha)|^2}d\beta d\alpha.$$

If we compute $\frac{1}{|\Delta z|^2}-\frac{1}{\beta^2|\partial_\alpha z(\alpha)|^2}=B_1+B_2+B_3$ where

$$B_1(\alpha,\beta)=\frac{\beta\int_0^1\int_0^1\frac{\partial_\alpha^2 z(\psi)-\partial_\alpha^2 z(\alpha)}{|\psi-\alpha|^\delta}\beta^\delta(1+s+t-st)^\delta(1-s)dtds\int_0^1[\partial_\alpha z(\alpha)+\partial_\alpha z(\phi)]ds}{|z(\alpha)-z(\alpha-\beta)|^2|\partial_\alpha z(\alpha)|^2},$$
$$B_3(\alpha,\beta)=\frac{\beta^2\partial_\alpha^2 z(\alpha)\int_0^1\int_0^1\partial_\alpha^2 z(\eta)(s-1)dtds}{|z(\alpha)-z(\alpha-\beta)|^2|\partial_\alpha z(\alpha)|^2},$$
$$B_4(\alpha,\beta)=\frac{\beta\partial_\alpha^2 z(\alpha)2\partial_\alpha z(\alpha)}{|z(\alpha)-z(\alpha-\beta)|^2|\partial_\alpha z(\alpha)|^2},$$

we can split $M_1=M_1^1+M_1^2+M_1^3$ for

$$M_1^1=\frac{\lambda_1}{\pi A(t)}\int_{\mathbb{T}}\int_{\mathbb{R}} H(\partial_\alpha^3 z\cdot\partial_\alpha^\perp z)(\alpha)\Delta\partial_\alpha^3 z^\perp\cdot\partial_\alpha z(\alpha)\varpi_1(\alpha-\beta)B_1(\alpha,\beta)d\beta d\alpha,$$
$$M_1^2=\frac{\lambda_1}{\pi A(t)}\int_{\mathbb{T}}\int_{\mathbb{R}} H(\partial_\alpha^3 z\cdot\partial_\alpha^\perp z)(\alpha)\Delta\partial_\alpha^3 z^\perp\cdot\partial_\alpha z(\alpha)\varpi_1(\alpha-\beta)B_2(\alpha,\beta)d\beta d\alpha,$$
$$M_1^3=\frac{\lambda_1}{\pi A(t)}\int_{\mathbb{T}}\int_{\mathbb{R}} H(\partial_\alpha^3 z\cdot\partial_\alpha^\perp z)(\alpha)\Delta\partial_\alpha^3 z^\perp\cdot\partial_\alpha z(\alpha)\varpi_1(\alpha-\beta)B_3(\alpha,\beta)d\beta d\alpha.$$

It is easy see that

$$M_1^1 \leq C\|\mathcal{F}(z)\|^2_{L^\infty}\|z\|_{\mathcal{C}^{2,\delta}}\|z\|_{\mathcal{C}^1}\|\varpi_1\|_{L^\infty}\|z\|^2_{H^3},$$
$$M_1^2 \leq C\|\mathcal{F}(z)\|^2_{L^\infty}\|z\|^2_{\mathcal{C}^2}\|\varpi_1\|_{L^\infty}\|z\|^2_{H^3}.$$

We need to study more precisely the term M_1^3. Again, we decompose M_1^3 in the following terms:

$$M_1^{31} = \frac{2\lambda_1}{\pi A^2(t)}\int_{\mathbb{T}}\int_{\mathbb{R}} H(\partial^3_\alpha z\cdot\partial^\perp_\alpha z)(\alpha)\Delta\partial^3_\alpha z^\perp\cdot\partial_\alpha z(\alpha)\varpi_1(\alpha-\beta)\beta\partial^2_\alpha z(\alpha)\cdot\partial_\alpha z(\alpha)B(\alpha,\beta)d\beta d\alpha,$$
$$M_1^{32} = \frac{2\lambda_1}{\pi A^3(t)}\int_{\mathbb{T}}\int_{\mathbb{R}} H(\partial^3_\alpha z\cdot\partial^\perp_\alpha z)(\alpha)\Delta\partial^3_\alpha z^\perp\cdot\partial_\alpha z(\alpha)\frac{\varpi_1(\alpha-\beta)}{\beta}\partial^2_\alpha z(\alpha)\cdot\partial_\alpha z(\alpha)d\beta d\alpha,$$

where

$$B(\alpha,\beta) = \frac{1}{|\Delta z|^2} - \frac{1}{\beta^2|\partial_\alpha z(\alpha)|^2} = \frac{\beta\int_0^1\int_0^1\partial^2_\alpha z(\psi)(1-s)dtds\cdot\int_0^1[\partial_\alpha z(\alpha)+\partial_\alpha z(\phi)]ds}{|z(\alpha)-z(\alpha-\beta)|^2|\partial_\alpha z(\alpha)|^2}.$$

Directly,

$$M_1^{31} \leq C\|\mathcal{F}(z)\|_{L^\infty}\|z\|^2_{\mathcal{C}^2}\|\varpi_1\|_{L^\infty}\|z\|_{H^3}.$$

For M_1^{32},

$$\begin{aligned}
M_1^{321} &= \frac{2\lambda_1}{\pi A^3(t)}\int_{\mathbb{T}}\int_{\mathbb{R}} H(\partial^3_\alpha z\cdot\partial^\perp_\alpha z)(\alpha)\partial^3_\alpha z(\alpha)^\perp\cdot\partial_\alpha z(\alpha)\frac{\varpi_1(\alpha-\beta)}{\beta}\partial^2_\alpha z(\alpha)\cdot\partial_\alpha z(\alpha)d\beta d\alpha\\
&= \frac{2\lambda_1}{\pi A^3(t)}\int_{\mathbb{T}} H(\partial^3_\alpha z\cdot\partial^\perp_\alpha z)(\alpha)\partial^3_\alpha z(\alpha)^\perp\cdot\partial_\alpha z(\alpha)H\varpi_1(\alpha)\partial^2_\alpha z(\alpha)\cdot\partial_\alpha z(\alpha)d\beta d\alpha\\
&\leq C\|\mathcal{F}(z)\|^{\frac{3}{2}}_{L^\infty}\|z\|_{\mathcal{C}^2}\|H\varpi_1\|_{L^\infty}\|z\|^2_{H^3}
\end{aligned}$$

and

$$\begin{aligned}
M_1^{322} &= \frac{2\lambda_1}{\pi A^3(t)}\int_{\mathbb{T}}\int_{\mathbb{R}} H(\partial^3_\alpha z\cdot\partial^\perp_\alpha z)(\alpha)\partial^3_\alpha z(\alpha-\beta)^\perp\cdot\partial_\alpha z(\alpha)\frac{\varpi_1(\alpha-\beta)}{\beta}\partial^2_\alpha z(\alpha)\cdot\partial_\alpha z(\alpha)d\beta d\alpha\\
&\leq C\|\mathcal{F}(z)\|^{\frac{3}{2}}_{L^\infty}\|z\|_{\mathcal{C}^2}\|\varpi\|_{\mathcal{C}^1}\|z\|^2_{H^3}\\
&+ \frac{2\lambda_1}{\pi A^3(t)}\int_{\mathbb{T}} H(\partial^3_\alpha z\cdot\partial^\perp_\alpha z)(\alpha)H(\partial^3_\alpha z^\perp)(\alpha)\cdot\partial_\alpha z(\alpha)\varpi_1(\alpha)\partial^2_\alpha z(\alpha)\cdot\partial_\alpha z(\alpha)d\alpha\\
&\leq C\|\mathcal{F}(z)\|^{\frac{3}{2}}_{L^\infty}\|z\|_{\mathcal{C}^2}\|\varpi\|_{\mathcal{C}^1}\|z\|^2_{H^3}.
\end{aligned}$$

Therefore, $M_1 \leq \exp C(\|\mathcal{F}(z)\|^2_{L^\infty}+\|d(z,h)\|^2_{L^\infty}+\|z\|^2_{H^3})$.

For M_2 we proceed as follows:

$$M_2^1 = \frac{\lambda_1}{\pi A^2(t)}\int_{\mathbb{T}}\int_{\mathbb{R}} H(\partial^3_\alpha z\cdot\partial^\perp_\alpha z)(\alpha)\Delta\partial^3_\alpha z^\perp\cdot\partial_\alpha z(\alpha)\frac{\varpi_1(\alpha-\beta)-\varpi_1(\alpha)}{\beta^2}d\beta d\alpha,$$
$$M_2^2 = \frac{\lambda_1}{\pi A^2(t)}\int_{\mathbb{T}} H(\partial^3_\alpha z\cdot\partial^\perp_\alpha z)(\alpha)\Lambda(\partial^3_\alpha z^\perp)(\alpha)\cdot\partial_\alpha z(\alpha)\varpi_1(\alpha)d\alpha.$$

In the same way as before,

$$
\begin{aligned}
M_2^1 &= \frac{\lambda_1}{\pi A^2(t)}\int_{\mathbb{T}} H(\partial_\alpha^3 z\cdot\partial_\alpha^\perp z)(\alpha)\partial_\alpha^3 z(\alpha)^\perp\cdot\partial_\alpha z(\alpha)\Lambda\varpi_1(\alpha)d\alpha\\
&+\frac{\lambda_1}{\pi A^2(t)}\int_{\mathbb{T}}\int_{\mathbb{R}} H(\partial_\alpha^3 z\cdot\partial_\alpha^\perp z)(\alpha)\partial_\alpha^3 z(\alpha-\beta)^\perp\cdot\partial_\alpha z(\alpha)\frac{\int_0^1\partial_\alpha\varpi_1(\alpha-\beta+\beta t)-\partial_\alpha\varpi_1(\alpha)dt}{\beta}d\beta d\alpha\\
&+\frac{\lambda_1}{\pi A^2(t)}\int_{\mathbb{T}} H(\partial_\alpha^3 z\cdot\partial_\alpha^\perp z)(\alpha)H(\partial_\alpha^3 z^\perp)(\alpha)\cdot\partial_\alpha z(\alpha)\partial_\alpha\varpi_1(\alpha)d\alpha\\
&\leq C\|\mathcal{F}(z)\|_{L^\infty}\|\varpi_1\|_{\mathcal{C}^{1,\delta}}\|z\|_{H^3}^2.
\end{aligned}
$$

Using the commutator estimation and $\Lambda H=-\partial_\alpha$

$$
\begin{aligned}
M_2^2 &= \frac{\lambda_1}{\pi A^2(t)}\int_{\mathbb{T}} H(\partial_\alpha^3 z\cdot\partial_\alpha^\perp z)(\alpha)(\Lambda(\partial_\alpha^3 z^\perp)(\alpha)\cdot\partial_\alpha z(\alpha)\varpi_1(\alpha)-\Lambda(\partial_\alpha^3 z^\perp\cdot\partial_\alpha z\varpi_1)(\alpha))d\alpha\\
&+\frac{\lambda_1}{\pi A^2(t)}\int_{\mathbb{T}} H(\partial_\alpha^3 z\cdot\partial_\alpha^\perp z)(\alpha)\Lambda(\partial_\alpha^3 z^\perp\cdot\partial_\alpha z\varpi_1)(\alpha)d\alpha\\
&\leq C\|\mathcal{F}(z)\|_{L^\infty}^2\|\partial_\alpha^3 z\partial_\alpha z\|_{L^2}\|\varpi_1\partial_\alpha z\|_{\mathcal{C}^{1,\delta}}\|\partial_\alpha^3 z\|_{L^2}\\
&-\frac{\lambda_1}{\pi A^2(t)}\int_{\mathbb{T}}\partial_\alpha(\partial_\alpha^3 z\cdot\partial_\alpha^\perp z)(\alpha)\partial_\alpha^3 z^\perp(\alpha)\cdot\partial_\alpha z(\alpha)\varpi_1(\alpha)d\alpha\\
&\leq \exp C(\|\mathcal{F}(z)\|_{L^\infty}^2+\|d(z,h)\|_{L^\infty}^2+\|z\|_{H^3}^2)\\
&+\frac{\lambda_1}{\pi A^2(t)}\int_{\mathbb{T}}\partial_\alpha(\partial_\alpha^3 z\cdot\partial_\alpha^\perp z)(\alpha)\partial_\alpha^3 z(\alpha)\cdot\partial_\alpha^\perp z(\alpha)\varpi_1(\alpha)d\alpha\\
&\leq \exp C(\|\mathcal{F}(z)\|_{L^\infty}^2+\|d(z,h)\|_{L^\infty}^2+\|z\|_{H^3}^2)\\
&+C\|\mathcal{F}(z)\|_{L^\infty}\|\varpi_1\|_{\mathcal{C}^1}\|z\|_{H^3}
\end{aligned}
$$

We can estimate L_3^{213} as before. Then, we get the estimation for L_3^{21}.

Let us estimate L_3^{22}. The most singular terms are

$$
\begin{aligned}
L_3^{221} &= \frac{\lambda_1}{\pi A(t)}\int_{\mathbb{T}} H(\partial_\alpha^3 z\cdot\partial_\alpha^\perp z)(\alpha)\partial_\alpha(BR(\partial_\alpha^2\varpi_2,h)_z)\cdot\partial_\alpha z(\alpha)d\alpha,\\
L_3^{222} &= \frac{\lambda_1}{\pi A(t)}\int_{\mathbb{T}}\int_{\mathbb{R}} H(\partial_\alpha^3 z\cdot\partial_\alpha^\perp z)(\alpha)\frac{(\Delta\partial_\alpha^3 zh)^\perp\cdot\partial_\alpha z(\alpha)}{|\Delta zh|^2}\varpi_2(\alpha-\beta)d\beta d\alpha\\
L_3^{223} &= \frac{\lambda_1}{\pi A(t)}\int_{\mathbb{T}}\int_{\mathbb{R}} H(\partial_\alpha^3 z\cdot\partial_\alpha^\perp z)(\alpha)\frac{(\Delta zh)^\perp\cdot\partial_\alpha z(\alpha)\Delta zh\cdot\Delta\partial_\alpha^3 zh}{|\Delta zh|^4}\varpi_2(\alpha-\beta)d\beta d\alpha
\end{aligned}
$$

Since

$$\partial_\alpha(BR(\partial_\alpha^2\varpi_2,h)_z)\cdot\partial_\alpha z=\partial_\alpha(T_2(\partial_\alpha^2\varpi_2))-BR(\partial_\alpha^2\varpi_2,h)_z\cdot\partial_\alpha^2 z$$

then,

$$L_3^{221}\leq C\|\mathcal{F}(z)\|_{L^\infty}\|\partial_\alpha^3 z\partial_\alpha^\perp z\|_{L^2}(\|T_2(\partial_\alpha^2\varpi_2)\|_{H^1}+\|BR(\partial_\alpha^2\varpi_2,h)_z\|_{L^2}\|\partial_\alpha^2 z\|_{L^\infty})$$

Using the estimation on $\|\mathcal{T}\varpi\|_{H^1}$ and $BR(\varpi_2,h)_z$, L_3^{221} is controlled. We can get,

$$L_3^{222}\leq C\|\mathcal{F}(z)\|_{L^\infty}\|d(z,h)\|_{L^\infty}\|\partial_\alpha^3 z\partial_\alpha z\|_{L^2}(\|\partial_\alpha^3 z\|_{L^2}+\|\partial_\alpha^3 h\|_{L^2})\|z\|_{\mathcal{C}^1}\|\varpi_2\|_{L^\infty}.$$

For L_3^{223} we get the same

$$L_3^{223} \leq C\|\mathcal{F}(z)\|_{L^\infty}\|d(z,h)\|_{L^\infty}\|\partial_\alpha^3 z\|_{L^2}(\|\partial_\alpha^3 z\|_{L^2}+\|\partial_\alpha^3 h\|_{L^2})\|z\|_{\mathcal{C}^1}^2\|\varpi_2\|_{L^\infty}.$$

For L_4 integrating by parts we obtain:

$$\begin{aligned}
&L_4 \leq C\|\mathcal{F}(z)\|_{L^\infty}\|\partial_\alpha^3 z\partial_\alpha z\|_{L^2}((\|\partial_\alpha^2 BR(\varpi_1,z)_z\|_{L^2}+\|\partial_\alpha^2 BR(\varpi_2,h)_z\|)\|\partial_\alpha^2 z\|_{L^\infty}\\
&+(\|\partial_\alpha BR(\varpi_1,z)_z\|_{L^\infty}+\|\partial_\alpha BR(\varpi_2,h)_z\|_{L^\infty})\|\partial_\alpha^3 z\|_{L^2})\\
&\leq \exp C(\|\mathcal{F}(z)\|_{L^\infty}^2+\|d(z,h)\|_{L^\infty}^2+\|z\|_{H^3}^2).
\end{aligned}$$

Finally we have to find $\sigma(\alpha)$ in L_5 to finish the estimations. To do that let us split $L_5 = L_5^1 + L_5^2 + L_5^3 + L_5^4$ where

$$\begin{aligned}
L_5^1 &= \frac{\lambda_1}{\pi A(t)}\int_{\mathbb{T}}\Lambda(\partial_\alpha^3 z_1\partial_\alpha z_2)(\alpha)(BR_1(\varpi_1,z)_z+BR_1(\varpi_2,h)_z)\partial_\alpha^3 z_1(\alpha)d\alpha,\\
L_5^2 &= \frac{\lambda_1}{\pi A(t)}\int_{\mathbb{T}}\Lambda(\partial_\alpha^3 z_1\partial_\alpha z_2)(\alpha)(BR_2(\varpi_1,z)_z+BR_2(\varpi_2,h)_z)\partial_\alpha^3 z_2(\alpha)d\alpha,\\
L_5^3 &= -\frac{\lambda_1}{\pi A(t)}\int_{\mathbb{T}}\Lambda(\partial_\alpha^3 z_2\partial_\alpha z_1)(\alpha)(BR_1(\varpi_1,z)_z+BR_1(\varpi_2,h)_z)\partial_\alpha^3 z_1(\alpha)d\alpha,\\
L_5^4 &= -\frac{\lambda_1}{\pi A(t)}\int_{\mathbb{T}}\Lambda(\partial_\alpha^3 z_2\partial_\alpha z_1)(\alpha)(BR_2(\varpi_1,z)_z+BR_2(\varpi_2,h)_z)\partial_\alpha^3 z_2(\alpha)d\alpha.
\end{aligned}$$

In order to reduce the notation, we denote $BR_i = BR_i(\varpi_1,z)_z + BR_i(\varpi_2,h)_z$ for $i=1,2$. Then we can write,

$$\begin{aligned}
&L_5^1 = \frac{\lambda_1}{\pi A(t)}\int_{\mathbb{T}}(\Lambda(\partial_\alpha^3 z_1\partial_\alpha z_2)(\alpha)-\partial_\alpha z_2\Lambda(\partial_\alpha^3 z)(\alpha))BR_1\partial_\alpha^3 z_1(\alpha)d\alpha\\
&+\frac{\lambda_1}{\pi A(t)}\int_{\mathbb{T}}BR_1\partial_\alpha z_2(\alpha)\partial_\alpha^3 z_1(\alpha)\Lambda(\partial_\alpha^3 z_1)(\alpha)d\alpha\\
&\leq C\|\partial_\alpha z\|_{\mathcal{C}^{1,\delta}}\|BR_1\|_{L^\infty}\|z\|_{H^3}^2+\frac{\lambda_1}{\pi A(t)}\int_{\mathbb{T}}BR_1\partial_\alpha z_2(\alpha)\partial_\alpha^3 z_1(\alpha)\Lambda(\partial_\alpha^3 z_1)(\alpha)d\alpha.
\end{aligned}$$

In the same way,

$$L_5^2 \leq \exp C(\|\mathcal{F}(z)\|_{L^\infty}^2+\|d(z,h)\|_{L^\infty}^2+\|z\|_{H^3}^2)+L_5^{21}$$

where

$$L_5^{21} = \frac{\lambda_1}{\pi A(t)}\int_{\mathbb{T}}BR_2\partial_\alpha z_2(\alpha)\partial_\alpha^3 z_2(\alpha)\Lambda(\partial_\alpha^3 z_1)(\alpha)d\alpha.$$

Using $\partial_\alpha z_2\partial_\alpha^3 z_2 = -\partial_\alpha z_1\partial_\alpha^3 z_1 - |\partial_\alpha^2 z|^2$ we separate L_5^{21} in

$$\begin{aligned}
L_5^{211} &= -\frac{\lambda_1}{\pi A(t)}\int_{\mathbb{T}}BR_2|\partial_\alpha^2 z(\alpha)|^2\Lambda(\partial_\alpha^3 z_1)(\alpha)d\alpha,\\
L_5^{212} &= -\frac{\lambda_1}{\pi A(t)}\int_{\mathbb{T}}BR_2\partial_\alpha z_1(\alpha)\partial_\alpha^3 z_1(\alpha)\Lambda(\partial_\alpha^3 z_1)(\alpha)d\alpha.
\end{aligned}$$

The fact that $\Lambda = \partial_\alpha H$ allows us to

$$
\begin{aligned}
L_5^{211} &= \frac{\lambda_1}{\pi A(t)} \int_{\mathbb{T}} \partial_\alpha (BR_2 |\partial_\alpha^2 z(\alpha)|^2) H(\partial_\alpha^3 z_1)(\alpha) d\alpha \\
&\leq C(\|\partial_\alpha BR_2\|_{L^2} \|z\|_{\mathcal{C}^2}^2 + \|BR_2\|_{L^2} \|z\|_{\mathcal{C}^2}^2) \|z\|_{H^3} \|\mathcal{F}(z)\|_{L^\infty} \\
&\leq \exp C(\|\mathcal{F}(z)\|_{L^\infty}^2 + \|d(z,h)\|_{L^\infty}^2 + \|z\|_{H^3}^2).
\end{aligned}
$$

Then we get,

$$
\begin{aligned}
L_5^2 &\leq \exp C(\|\mathcal{F}(z)\|_{L^\infty}^2 + \|d(z,h)\|_{L^\infty}^2 + \|z\|_{H^3}^2) \\
&- \frac{\lambda_1}{\pi A(t)} \int_{\mathbb{T}} BR_2 \partial_\alpha z_1(\alpha) \partial_\alpha^3 z_1(\alpha) \Lambda(\partial_\alpha^3 z_1)(\alpha) d\alpha.
\end{aligned}
$$

Now, we add $L_5^1 + L_5^2$:

$$
\begin{aligned}
L_5^1 + L_5^2 &\leq \exp C(\|\mathcal{F}(z)\|_{L^\infty}^2 + \|d(z,h)\|_{L^\infty}^2 + \|z\|_{H^3}^2) \\
&- \frac{\lambda_1}{\pi A(t)} \int_{\mathbb{T}} (BR(\varpi_1, z)_z + BR(\varpi_2, h)_z) \cdot \partial_\alpha^\perp z(\alpha) \partial_\alpha^3 z_1(\alpha) \Lambda(\partial_\alpha^3 z_1)(\alpha) d\alpha.
\end{aligned}
$$

Analogously, using $\partial_\alpha z_1 \partial_\alpha^3 z_1 = -\partial_\alpha z_2 \partial_\alpha^3 z_2 - |\partial_\alpha^2 z|^2$ we get:

$$
\begin{aligned}
L_5^3 + L_5^4 &\leq \exp C(\|\mathcal{F}(z)\|_{L^\infty}^2 + \|d(z,h)\|_{L^\infty}^2 + \|z\|_{H^3}^2) \\
&- \frac{\lambda_1}{\pi A(t)} \int_{\mathbb{T}} (BR(\varpi_1, z)_z + BR(\varpi_2, h)_z) \cdot \partial_\alpha^\perp z(\alpha) \partial_\alpha^3 z_2(\alpha) \Lambda(\partial_\alpha^3 z_2)(\alpha) d\alpha.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
L_5 &\leq \exp C(\|\mathcal{F}(z)\|_{L^\infty}^2 + \|d(z,h)\|_{L^\infty}^2 + \|z\|_{H^3}^2) \\
&- \frac{\lambda_1}{\pi A(t)} \int_{\mathbb{T}} (BR(\varpi_1, z)_z + BR(\varpi_2, h)_z) \cdot \partial_\alpha^\perp z(\alpha) \partial_\alpha^3 z(\alpha) \cdot \Lambda(\partial_\alpha^3 z)(\alpha) d\alpha.
\end{aligned}
$$

In conclusion,

$$
\begin{aligned}
I_1^4 &\leq \exp C(\|\mathcal{F}(z)\|_{L^\infty}^2 + \|d(z,h)\|_{L^\infty}^2 + \|z\|_{H^3}^2) \\
&- \frac{1}{\pi A(t)} \int_{\mathbb{T}} \Big[\lambda_1 (BR(\varpi_1, z)_z + BR(\varpi_2, h)_z) \cdot \partial_\alpha^\perp z(\alpha) + N \partial_\alpha z_1(\alpha)\Big] \partial_\alpha^3 z(\alpha) \cdot \Lambda(\partial_\alpha^3 z)(\alpha) d\alpha.
\end{aligned}
$$

Since,

$$
\sigma(\alpha, t) = \frac{\mu_2 - \mu_1}{\kappa_1} (BR(\varpi_1, z)_z + BR(\varpi_2, h)_z) \cdot \partial_\alpha^\perp z(\alpha) + (\rho_2 - \rho_1) g \partial_\alpha z_1(\alpha)
$$

then,

$$
\begin{aligned}
I_1 &\leq \exp C(\|\mathcal{F}(z)\|_{L^\infty}^2 + \|d(z,h)\|_{L^\infty}^2 + \|z\|_{H^3}^2) \\
&- \frac{\kappa_1}{2\pi(\mu_2 + \mu_1) A(t)} \int_{\mathbb{T}} \sigma(\alpha, t) \partial_\alpha^3 z(\alpha) \cdot \Lambda(\partial_\alpha^3 z)(\alpha) d\alpha.
\end{aligned}
$$

2.5.2 Estimates on I_3

To finish all estimation on z, we consider:

$$\begin{aligned}I_3 = &\int_{\mathbb{T}} \partial_\alpha^3 z(\alpha)\cdot\partial_\alpha^4 z(\alpha)c(\alpha)d\alpha + 3\int_{\mathbb{T}}|\partial_\alpha^3 z(\alpha)|^2\partial_\alpha c(\alpha)d\alpha\\ &+3\int_{\mathbb{T}}\partial_\alpha^3 z(\alpha)\cdot\partial_\alpha^2 z(\alpha)\partial_\alpha^2 c(\alpha)d\alpha + \int_{\mathbb{T}}\partial_\alpha^3 z(\alpha)\cdot\partial_\alpha z(\alpha)\partial_\alpha^3 c(\alpha)d\alpha\\ &I_3^1+I_3^2+I_3^3+I_3^4.\end{aligned}$$

Integrating by parts and using the definition of $c(\alpha)$,

$$\begin{aligned}I_3^1 &= -\frac{1}{2}\int_{\mathbb{T}}|\partial_\alpha^3 z(\alpha)|^2\partial_\alpha c(\alpha)d\alpha \leq C\|\partial_\alpha c\|_{L^\infty}\|\partial_\alpha^3 z\|_{L^2}^2\\ &\leq 2C\|\mathcal{F}(z)\|_{L^\infty}^{\frac{1}{2}}(\|\partial_\alpha BR(\varpi_1,z)_z\|_{L^\infty}+\|\partial_\alpha BR(\varpi_2,h)_z\|_{L^\infty})\|\partial_\alpha^3 z\|_{L^2}^2.\end{aligned}$$

Since $I_3^2 = -6I_3^1$ we have I_3^2 controlled. Computing $\partial_\alpha^2 c$,

$$\begin{aligned}\partial_\alpha^2 c(\alpha) = &-\frac{\partial_\alpha^2 z(\alpha)}{A(t)}\cdot(\partial_\alpha BR(\varpi_1,z)_z+\partial_\alpha BR(\varpi_2,h)_z)\\ &-\frac{\partial_\alpha z(\alpha)}{A(t)}\cdot(\partial_\alpha^2 BR(\varpi_1,z)_z+\partial_\alpha^2 BR(\varpi_2,h)_z).\end{aligned}$$

Thus,

$$\begin{aligned}I_3^3 &= -3\int_{\mathbb{T}}\partial_\alpha^3 z(\alpha)\cdot\partial_\alpha^2 z(\alpha)\frac{\partial_\alpha^2 z(\alpha)}{A(t)}\cdot(\partial_\alpha BR(\varpi_1,z)_z+\partial_\alpha BR(\varpi_2,h)_z)d\alpha\\ &-3\int_{\mathbb{T}}\partial_\alpha^3 z(\alpha)\cdot\partial_\alpha^2 z(\alpha)\frac{\partial_\alpha z(\alpha)}{A(t)}\cdot(\partial_\alpha^2 BR(\varpi_1,z)_z+\partial_\alpha^2 BR(\varpi_2,h)_z)d\alpha\\ &\equiv I_3^{31}+I_3^{32}\end{aligned}$$

where

$$\begin{aligned}I_3^{31} &\leq C\|\mathcal{F}(z)\|_{L^\infty}\|z\|_{\mathcal{C}^2}^2(\|\partial_\alpha BR(\varpi_1,z)_z\|_{L^\infty}+\|\partial_\alpha BR(\varpi_2,h)_z\|_{L^\infty})\|\partial_\alpha^3 z\|_{L^2},\\ I_3^{32} &\leq C\|\mathcal{F}(z)\|_{L^\infty}^{\frac{1}{2}}\|z\|_{\mathcal{C}^2}(\|\partial_\alpha^2 BR(\varpi_1,z)_z\|_{L^2}+\|\partial_\alpha^2 BR(\varpi_2,h)_z\|_{L^2})\|\partial_\alpha^3 z\|_{L^2}.\end{aligned}$$

Using the estimation on $\|BR(\varpi_1,z)_z\|_{H^k}+\|BR(\varpi_2,h)_z\|_{H^k}$ we obtain,

$$I_3^3 \leq \exp C(\|\mathcal{F}(z)\|_{L^\infty}^2+\|d(z,h)\|_{L^\infty}^2+\|z\|_{H^3}^2).$$

Since $|\partial_\alpha z(\alpha)|^2 = A(t)$, if we differenciate respect to α:

$$0 = 2|\partial_\alpha^2 z(\alpha)|^2+2\partial_\alpha z(\alpha)\cdot\partial_\alpha^3 z(\alpha) \Rightarrow \partial_\alpha z(\alpha)\cdot\partial_\alpha^3 z(\alpha) = -|\partial_\alpha^2 z(\alpha)|^2.$$

Then, integrating by parts

$$I_3^4 = -\int_{\mathbb{T}}|\partial_\alpha^2 z(\alpha)|^2\partial_\alpha^3 c(\alpha)d\alpha = 2\int_{\mathbb{T}}\partial_\alpha^2 z(\alpha)\cdot\partial_\alpha^3 z(\alpha)\partial_\alpha^2 c(\alpha)d\alpha = \frac{2}{3}I_3^3.$$

Therefore,

$$I_3^4 \leq \exp C(\|\mathcal{F}(z)\|_{L^\infty}^2 + \|d(z,h)\|_{L^\infty}^2 + \|z\|_{H^3}^2).$$

Putting together all above estimates, since the case for $k > 3$ is straightforward we have

$$\frac{d}{dt}\|z\|_{H^k}^2 \leq \exp C(\|\mathcal{F}(z)\|_{L^\infty}^2 + \|d(z,h)\|_{L^\infty}^2 + \|z\|_{H^k}^2)$$
$$- \frac{\kappa^1}{2\pi(\mu^1+\mu^2)} \int_{\mathbb{T}} \frac{\sigma(\alpha)}{A(t)} \partial_\alpha^k z(\alpha) \cdot \Lambda(\partial_\alpha^k z)(\alpha) d\alpha$$

for $k \geq 3$.

2.6 Evolution of the arc-chord condition

Lemma 2.6.1. *The following estimate holds*

$$\frac{d}{dt}\|\mathcal{F}(z)\|_{L^\infty}^2(t) \leq \exp C(\|\mathcal{F}(z)\|_{L^\infty}^2 + \|d(z,h)\|_{L^\infty}^2 + \|z\|_{H^3}^2).$$

Proof. Let take $p > 1$. It follows that

$$\frac{d}{dt}\|\mathcal{F}(z)\|_{L^p}^p(t) = \frac{d}{dt}\int_{\mathbb{R}}\int_{\mathbb{R}} (\frac{\beta^2}{|z(\alpha)-z(\alpha-\beta)|^2})^p d\alpha d\beta$$
$$- 2p\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{\beta^{2p}\Delta z \cdot \Delta z_t}{|\Delta z|^{2p+2}} d\alpha d\beta = -2p\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{\beta^{2p+1}\int_0^1 \partial_\alpha z(\alpha-\beta+\beta t)dt \cdot \Delta z_t}{|\Delta z|^{2p+2}} d\alpha d\beta.$$

Since,

$$z_t(\alpha) - z_t(\alpha-\beta) = ((BR(\varpi_1,z)_z(\alpha) - BR(\varpi_1,z)_z(\alpha-\beta))$$
$$+ (BR(\varpi_2,h)_z(\alpha) - BR(\varpi_2,h)_z(\alpha-\beta))) + (c(\alpha)-c(\alpha-\beta))\partial_\alpha z(\alpha-\beta)$$
$$+ c(\alpha-\beta)(\partial_\alpha z(\alpha) - \partial_\alpha z(\alpha-\beta)) \equiv J_1 + J_2 + J_3$$

we have,

$$J_1 = \beta\int_0^1 \partial_\alpha BR(\varpi_1,z)_z(\alpha-\beta+t\beta) + \partial_\alpha BR(\varpi_2,h)_z(\alpha-\beta+t\beta)dt \tag{2.21}$$
$$\leq |\beta|(\|\partial_\alpha BR(\varpi_1,z)_z\|_{L^\infty} + \|\partial_\alpha BR(\varpi_2,h)_z\|_{L^\infty}),$$

$$J_2 \leq \frac{|\beta|}{A^{\frac{1}{2}}(t)}(\|\partial_\alpha BR(\varpi_1,z)_z\|_{L^\infty} + \|\partial_\alpha BR(\varpi_2,h)_z\|_{L^\infty}) \tag{2.22}$$

and

$$J_3 = c(\alpha-\beta)\beta\int_0^1 \partial_\alpha^2 z(\alpha-\beta+t\beta)dt \leq \|c\|_{L^\infty}|\beta|\|z\|_{\mathcal{C}^2}. \tag{2.23}$$

Then,

$$\frac{d}{dt}\|\mathcal{F}(z)\|_{L^p}^p(t) \leq 2p\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{\beta^{2p+2}}{|\Delta z|^{2p+2}}\int_0^1 \partial_\alpha z(\alpha-\beta+\beta t)dt \cdot (\frac{J_1+J_2+J_3}{|\beta|}) d\alpha d\beta$$

$$\leq 2p \exp C|||z,h|||^2 \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{\beta^{2p+2}}{|\Delta z|^{2p+2}} d\alpha d\beta \leq p \exp C|||z,h|||^2 \|\mathcal{F}(z)\|_{L^\infty} \|\mathcal{F}(z)\|_{L^p}^p$$

and

$$\frac{d}{dt}\|\mathcal{F}(z)\|_{L^p}^p(t) \leq \exp C|||z,h|||^2 \|\mathcal{F}(z)\|_{L^p}(t)$$

Let integrate on t,

$$\|\mathcal{F}(z)\|_{L^p}(t+h) \leq \|\mathcal{F}(z)\|_{L^p}(t)$$

If we take $p \to \infty$ we get

$$\|\mathcal{F}(z)\|_{L^\infty}(t+h) \leq \|\mathcal{F}(z)\|_{L^\infty}(t) \exp(\int_t^{t+h} \exp C|||z,h|||^2(s)ds),$$

then

$$\begin{aligned}
&\frac{d}{dt}\|\mathcal{F}(z)\|_{L^\infty}(t) = \lim_{h\to 0}(\frac{\|\mathcal{F}(z)\|_{L^\infty}(t+h) - \|\mathcal{F}(z)\|_{L^\infty}(t)}{h})\\
&\leq \|\mathcal{F}(z)\|_{L^\infty}(t) \lim_{h\to 0}(\frac{\exp\int_t^{t+h}\exp|||z,h|||^2(s)ds - 1}{h})\\
&\leq \|\mathcal{F}(z)\|_{L^\infty}(t) \exp|||z,h|||^2(t).
\end{aligned}$$

■

2.7 Evolution of the distance between z and h

Remind that we relate the distance of the curve z with h through the function

$$d(z,h) = \frac{1}{|z(\alpha) - h(\alpha-\beta)|^2}$$

Lemma 2.7.1. *The following estimate holds*

$$\frac{d}{dt}\|d(z,h)\|_{L^\infty}^2 \leq \exp C(\|\mathcal{F}(z)\|_{L^\infty}^2 + \|d(z,h)\|_{L^\infty}^2 + \|z\|_{H^3}^2).$$

Proof. If we take $p > 1$, we get

$$\begin{aligned}
&\frac{d}{dt}\|d(z,h)\|_{L^p}^p(t) = \frac{d}{dt}\int_{\mathbb{T}}\int_{\mathbb{T}} \frac{1}{|z(\alpha)-h(\alpha-\beta)|^{2p}} d\alpha d\beta\\
&= -2p\int_{\mathbb{T}}\int_{\mathbb{T}} \frac{(z(\alpha)-h(\alpha-\beta))\cdot z_t(\alpha)}{|z(\alpha)-h(\alpha-\beta)|^{2p+2}} d\alpha d\beta\\
&= -2p\int_{\mathbb{T}}\int_{\mathbb{T}} \frac{z(\alpha)\cdot z_t(\alpha)}{|z(\alpha)-h(\alpha-\beta)|^{2p+2}} d\alpha d\beta + 2p\int_{\mathbb{T}}\int_{\mathbb{T}} \frac{h(\alpha-\beta)\cdot z_t(\alpha)}{|z(\alpha)-h(\alpha-\beta)|^{2p+2}} d\alpha d\beta\\
&= -2p\int_{\mathbb{T}}\int_{\mathbb{T}} \frac{z(\alpha)\cdot(BR(\varpi_1,z)_z + BR(\varpi_2,h)_z)}{|z(\alpha)-h(\alpha-\beta)|^{2p+2}} d\alpha d\beta\\
&- 2p\int_{\mathbb{T}}\int_{\mathbb{T}} \frac{z(\alpha\cdot\partial_\alpha z(\alpha)c(\alpha)}{|z(\alpha)-h(\alpha-\beta)|^{2p+2}} d\alpha d\beta + 2p\int_{\mathbb{T}}\int_{\mathbb{T}} \frac{h(\alpha-\beta)\cdot(BR(\varpi_1,z)_z + BR(\varpi_2,h)_z)}{|z(\alpha)-h(\alpha-\beta)|^{2p+2}} d\alpha d\beta\\
&+ 2p\int_{\mathbb{T}}\int_{\mathbb{T}} \frac{h(\alpha-\beta)\cdot\partial_\alpha z(\alpha)c(\alpha)}{|z(\alpha)-h(\alpha-\beta)|^{2p+2}} d\alpha d\beta \equiv J_1 + J_2 + J_3 + J_4.
\end{aligned}$$

It is easy to see that

$$\begin{aligned}
J_1 &\leq C\|d(z,h)\|_{L^\infty}\|z\|_{L^2}\|BR(\varpi_1,z)_z+BR(\varpi_2,h)_z\|_{L^2}\int_{\mathbb{T}}\int_{\mathbb{T}}\frac{1}{|z(\alpha)-h(\alpha-\beta)|^{2p}}d\alpha d\beta\\
&\leq \exp C(\|\mathcal{F}(z)\|^2_{L^\infty}+\|d(z,h)\|^2_{L^\infty}+\|z\|^2_{H^3})\|d(z,h)\|^p_{L^p},\\
J_3 &\leq C\|d(z,h)\|_{L^\infty}\|h\|_{L^2}\|BR(\varpi_1,z)_z+BR(\varpi_2,h)_z\|_{L^2}\int_{\mathbb{T}}\int_{\mathbb{T}}\frac{1}{|z(\alpha)-h(\alpha-\beta)|^{2p}}d\alpha d\beta\\
&\leq \exp C(\|\mathcal{F}(z)\|^2_{L^\infty}+\|d(z,h)\|^2_{L^\infty}+\|z\|^2_{H^3})\|d(z,h)\|^p_{L^p},\\
J_2 &\leq C\|d(z,h)\|_{L^\infty}\|c\|_{L^\infty}\|z\|_{L^\infty}\|z\|_{\mathcal{C}^1}\int_{\mathbb{T}}\int_{\mathbb{T}}\frac{1}{|z(\alpha)-h(\alpha-\beta)|^{2p}}d\alpha d\beta\\
&\leq \exp C(\|\mathcal{F}(z)\|^2_{L^\infty}+\|d(z,h)\|^2_{L^\infty}+\|z\|^2_{H^3})\|d(z,h)\|^p_{L^p},
\end{aligned}$$

and

$$\begin{aligned}
J_4 &\leq C\|d(z,h)\|_{L^\infty}\|c\|_{L^\infty}\|h\|_{L^\infty}\|z\|_{\mathcal{C}^1}\int_{\mathbb{T}}\int_{\mathbb{T}}\frac{1}{|z(\alpha)-h(\alpha-\beta)|^{2p}}d\alpha d\beta\\
&\leq \exp C(\|\mathcal{F}(z)\|^2_{L^\infty}+\|d(z,h)\|^2_{L^\infty}+\|z\|^2_{H^3})\|d(z,h)\|^p_{L^p}.
\end{aligned}$$

Therefore,

$$\frac{d}{dt}\|d(z,h)\|^p_{L^p}\leq \exp C|||z,h|||^2\|d(z,h)\|^p_{L^p}.$$

Let integrate on t,

$$\|d(z,h)\|_{L^p}(t+h)\leq\|d(z,h)\|_{L^p}(t)\exp(\int_t^{t+h}\exp C|||z,h|||^2(s)ds).$$

If we take $p\to\infty$ we get

$$\|d(z,h)\|_{L^\infty}(t+h)\leq\|d(z,h)\|_{L^\infty}(t)\exp(\int_t^{t+h}\exp C|||z,h|||^2(s)ds),$$

then

$$\begin{aligned}
\frac{d}{dt}\|d(z,h)\|_{L^\infty}(t) &= \lim_{h\to 0}(\frac{\|d(z,h)\|_{L^\infty}(t+h)-\|d(z,h)\|_{L^\infty}(t)}{h})\\
&\leq \|d(z,h)\|_{L^\infty}(t)\lim_{h\to 0}(\frac{\exp\int_t^{t+h}\exp|||z,h|||^2(s)ds-1}{h})\\
&\leq \|d(z,h)\|_{L^\infty}(t)\exp|||z,h|||^2(t).
\end{aligned}$$

■

2.8 Evolution of the minimum of $\sigma(\alpha,t)$

We know that

$$\sigma(\alpha,t)=\frac{\mu^2-\mu^1}{\kappa^1}(BR(\varpi_1,z)_z+BR(\varpi_2,h)_z)\cdot\partial_\alpha^\perp z(\alpha)+(\rho^2-\rho^1)g\partial_\alpha z_1(\alpha).$$

Lemma 2.8.1. *Let $z(\alpha,t)$ be a solution of the system with $z(\alpha,t)\in\mathcal{C}^1([0,T];H^3)$ and $m(t)=$*

$\min_{\alpha\in\mathbb{T}}\sigma(\alpha,t)$. *Then*

$$m(t)\geq m(0)-\int_0^t \exp C|||z,h|||^2(s)ds.$$

Recall that

$$\exp C|||z,h|||^2=\exp C(\|\mathcal{F}(z)\|^2_{L^\infty}+\|d(z,h)\|^2_{L^\infty}+\|z\|^2_{H^3}).$$

Proof. We consider $\alpha_t\in\mathbb{T}$ such that

$$m(t)=\min_{\alpha\in\mathbb{T}}\sigma(\alpha,t)=\sigma(\alpha_t,t).$$

We may calculate the derivate of $m(t)$, to obtain

$$m'(t)=\sigma_t(\alpha_t,t).$$

Using the definition,

$$\begin{aligned}
&\sigma_t(\alpha,t)=\frac{\mu^2-\mu^1}{\kappa^1}(\partial_t BR(\varpi_1,z)_z+\partial_t BR(\varpi_2,h)_z)\cdot\partial_\alpha^\perp z(\alpha)\\
&+(\frac{\mu^2-\mu^1}{\kappa^1}(BR(\varpi_1,z)_z+BR(\varpi_2,h)_z)\cdot\partial_\alpha^\perp z_t(\alpha)+(\rho^2-\rho^1)g\partial_\alpha\partial_t z_1(\alpha))\\
&\equiv I_1+I_2.
\end{aligned}$$

We have,

$$\begin{aligned}
|I_2|&\leq C(\|BR(\varpi_1,z)_z+BR(\varpi_2,h)_z\|_{L^\infty}+1)\|\partial_\alpha z_t\|_{L^\infty}\\
&\leq\exp C|||z,h|||^2\|\partial_\alpha z_t\|_{L^\infty}.
\end{aligned}$$

Using the equation of z_t we can calculate the estimations of $\|\partial_\alpha z_t\|_{L^\infty}$. We have,

$$\begin{aligned}
&\|\partial_\alpha z_t\|_{L^\infty}\leq\|BR(\varpi_1,z)_z\|_{L^\infty}+\|BR(\varpi_2,h)_z\|_{L^\infty}+\|\partial_\alpha c\|_{L^\infty}\|\partial_\alpha z\|_{L^\infty}\\
&+\|c\|_{L^\infty}\|\partial^2_\alpha z\|_{L^\infty}\leq\exp C|||z,h|||^2
\end{aligned}\tag{2.24}$$

then we obtain

$$|I_2|\leq\exp C|||z,h|||^2.$$

Let us write $\partial_t BR(\varpi_1,z)_z=B_1+B_2+B_3$ where

$$\begin{aligned}
B_1&=\frac{1}{2\pi}PV\int_{\mathbb{R}}\frac{(\Delta z)^\perp}{|\Delta z|^2}\partial_t\varpi_1(\alpha-\beta)d\alpha d\beta,\\
B_2&=\frac{1}{2\pi}PV\int_{\mathbb{R}}\frac{(\Delta z_t)^\perp}{|\Delta z|^2}\varpi_1(\alpha-\beta)d\alpha d\beta,\\
B_3&=-\frac{1}{\pi}PV\int_{\mathbb{R}}\frac{(\Delta z)^\perp\Delta z\cdot\Delta z_t}{|\Delta z|^4}\varpi_1(\alpha-\beta)d\alpha d\beta.
\end{aligned}$$

We split B_1 in the following way,

$$B_1=\frac{1}{2\pi}PV\int_{\mathbb{R}}(\frac{(\Delta z)^{\perp}}{|\Delta z|^2}-\frac{\partial_\alpha^{\perp}z(\alpha)}{\beta|\partial_\alpha z(\alpha)|^2})\partial_t\varpi_1(\alpha-\beta)d\alpha d\beta+\frac{\partial_\alpha^{\perp}z(\alpha)}{|\partial_\alpha z(\alpha)|^2}H(\partial_t\varpi_1)(\alpha).$$

Then,

$$|B_1|\leq C\|\mathcal{F}(z)\|_{L^\infty}\|z\|_{\mathcal{C}^2}\|\partial_t\varpi_1\|_{L^2}+\|\mathcal{F}(z)\|_{L^\infty}^{\frac{1}{2}}\|\partial_t\varpi_1\|_{\mathcal{C}^\delta}.$$

For estimate B_2 we separate

$$\begin{aligned}B_2&=\frac{1}{2\pi}PV\int_{\mathbb{R}}(\Delta z_t)^{\perp}(\frac{1}{|\Delta z|^2}-\frac{1}{\beta^2|\partial_\alpha z(\alpha)|^2})\varpi_1(\alpha-\beta)d\alpha d\beta\\&+\frac{1}{2\pi}PV\int_{\mathbb{R}}\frac{(\Delta z_t)^{\perp}}{\beta^2|\partial_\alpha z(\alpha)|^2}\varpi_1(\alpha-\beta)d\alpha d\beta\equiv B_2^1+B_2^2.\end{aligned}$$

Computing $\frac{1}{|\Delta z|^2}-\frac{1}{\beta^2|\partial_\alpha z(\alpha)|^2}$ and using (2.21), (2.22) and (2.23), we get

$$|B_2^1|\leq C\exp C|||z,h|||^2\|\mathcal{F}(z)\|_{L^\infty}^{\frac{3}{2}}\|z\|_{H^2}\|\varpi_1\|_{L^2}.$$

Since,

$$\Delta z_t=\beta^2\int_0^1\int_0^1\partial_\alpha^2 z(\alpha-\beta s+\beta ts)(1-t)dsdt+\beta\partial_\alpha z_t(\alpha)$$

then

$$B_2^2\leq C\|\mathcal{F}(z)\|_{L^\infty}\|\partial_\alpha^2 z_t\|_{L^2}\|\varpi_1\|_{L^2}+C\|\mathcal{F}(z)\|_{L^\infty}\|\partial_\alpha z_t\|_{L^\infty}\|H(\varpi_1)\|_{L^2}.$$

Using (2.24) and

$$\begin{aligned}\|\partial_\alpha^2 z_t\|_{L^2}&\leq\|\partial_\alpha^2 BR(\varpi_1,z)_z+\partial_\alpha^2 BR(\varpi_2,h)_z\|_{L^2}+\|\partial_\alpha^2 c\|_{L^2}\|\partial_\alpha z\|_{L^\infty}\\&+\|\partial_\alpha c\|_{L^\infty}\|\partial_\alpha^2 z\|_{L^2}+\|c\|_{L^\infty}\|\partial_\alpha^3 z\|_{L^2},\end{aligned}\tag{2.25}$$

since

$$\begin{aligned}\partial_\alpha^2 c(\alpha)=&-\frac{\partial_\alpha^2 z(\alpha)}{A(t)}(\partial_\alpha BR(\varpi_1,z)_z+\partial_\alpha BR(\varpi_2,h)_z)\\&-\frac{\partial_\alpha z(\alpha)}{A(t)}(\partial_\alpha^2 BR(\varpi_1,z)_z+\partial_\alpha^2 BR(\varpi_2,h)_z),\end{aligned}$$

we get

$$B_2^2\leq\exp C|||z,h|||^2.$$

Using the same proceeding, we have $B_3\leq\exp C|||z,h|||^2$.

On the other hand, we split $\partial_t BR(\varpi_2,h)_z=C_1+C_2+C_3$ where,

$$C_1=\frac{1}{2\pi}PV\int_{\mathbb{R}}\frac{(\Delta zh)^{\perp}}{|\Delta zh|^2}\partial_t\varpi_2(\alpha-\beta)d\alpha d\beta,$$

$$C_2 = \frac{1}{2\pi} PV \int_{\mathbb{R}} \frac{\partial_t^{\perp} z(\alpha)}{|\Delta zh|^2} \varpi_2(\alpha-\beta) d\alpha d\beta,$$
$$C_3 = -\frac{1}{\pi} PV \int_{\mathbb{R}} \frac{(\Delta zh)^{\perp} \Delta zh \cdot \partial_t z(\alpha)}{|\Delta z|^4} \varpi_2(\alpha-\beta) d\alpha d\beta.$$

Thus we have

$$C_1 \leq C\|d(z,h)\|_{L^\infty}^{\frac{1}{2}} \|\partial_t \varpi_2\|_{L^2},$$
$$C_2 \leq C\|d(z,h)\|_{L^\infty} \|\partial_t z\|_{L^\infty} \|\varpi_2\|_{L^2} \leq \exp C|||z,h|||^2,$$
$$C_3 \leq C\|d(z,h)\|_{L^\infty} \|\partial_\alpha z_t\|_{L^\infty} \|\varpi_2\|_{L^2} \leq \exp C|||z,h|||^2.$$

We only need to know what happen with $\|\partial_t \varpi_1\|_{L^2}$, $\|\partial_t \varpi_2\|_{L^2}$ and $\|\varpi_1\|_{\mathcal{C}^\delta}$. Using the definitions of $\partial_t \varpi_1$ and $\partial_t \varpi_2$ we can see that

$$\varpi_t + M\mathcal{T}(\varpi_t) = -M\mathbb{R}\varpi - \begin{pmatrix} N\partial_t \partial_\alpha z_2(\alpha) \\ 0 \end{pmatrix}$$

where

$$\mathbb{R} = \begin{pmatrix} R_1 & R_2 \\ R_3 & 0 \end{pmatrix}$$

with

$$R_1(\varpi_1) = \frac{1}{\pi} PV \int_{\mathbb{R}} \partial_t (\frac{(z(\alpha)-z(\alpha-\beta))^{\perp} \cdot \partial_\alpha z(\alpha)}{|z(\alpha)-z(\alpha-\beta)|^2}) \varpi_1(\alpha-\beta) d\beta,$$
$$R_2(\varpi_2) = \frac{1}{\pi} PV \int_{\mathbb{R}} \partial_t (\frac{(z(\alpha)-h(\alpha-\beta))^{\perp} \cdot \partial_\alpha z(\alpha)}{|z(\alpha)-h(\alpha-\beta)|^2}) \varpi_2(\alpha-\beta) d\beta,$$
$$R_3(\varpi_1) = \frac{1}{\pi} PV \int_{\mathbb{R}} \partial_t (\frac{(h(\alpha)-z(\alpha-\beta))^{\perp} \cdot \partial_\alpha z(\alpha)}{|h(\alpha)-z(\alpha-\beta)|^2}) \varpi_1(\alpha-\beta) d\beta.$$

Then,

$$\|\varpi_t\|_{H^{\frac{1}{2}}} \leq \|(I+M\mathcal{T})^{-1}\|_{H^{\frac{1}{2}}} (\|\mathbb{R}\varpi\|_{H^{\frac{1}{2}}} + \|\partial_\alpha z_t\|_{H^{\frac{1}{2}}}).$$

Therefore, it is clear that in order to control $\|\varpi_t\|_{L^2}$ we only need to estimate $\|\mathbb{R}\varpi\|_{H^{\frac{1}{2}}}$. To do that, let us estimate $\|\mathbb{R}\varpi\|_{H^1}$:

Let separates $R_1(\varpi_1) = S_1 + S_2 + S_3$ where

$$S_1 = \frac{1}{\pi} PV \int_{\mathbb{R}} \frac{(z_t(\alpha)-z_t(\alpha-\beta))^{\perp} \cdot \partial_\alpha z(\alpha)}{|z(\alpha)-z(\alpha-\beta)|^2} \varpi_1(\alpha-\beta) d\beta,$$
$$S_2 = \frac{1}{\pi} PV \int_{\mathbb{R}} \frac{(z(\alpha)-z(\alpha-\beta))^{\perp} \cdot \partial_\alpha z_t(\alpha)}{|z(\alpha)-z(\alpha-\beta)|^2} \varpi_1(\alpha-\beta) d\beta,$$
$$S_3 = -\frac{2}{\pi} PV \int_{\mathbb{R}} \frac{(\Delta z)^{\perp} \cdot \partial_\alpha z(\alpha) \Delta z \cdot \Delta z_t}{|z(\alpha)-z(\alpha-\beta)|^4} \varpi_1(\alpha-\beta) d\beta.$$

We will estimate $\partial_\alpha S_1$, the other terms $\partial_\alpha S_2$ and $\partial_\alpha S_3$ are estimated with the same procedure.

$$\begin{aligned}\partial_\alpha S_1 &= \frac{1}{\pi}PV\int_{\mathbb{R}}\frac{(\partial_\alpha z_t(\alpha)-\partial_\alpha z_t(\alpha-\beta))^\perp\cdot\partial_\alpha z(\alpha)}{|z(\alpha)-z(\alpha-\beta)|^2}\varpi_1(\alpha-\beta)d\beta\\
&+\frac{1}{\pi}PV\int_{\mathbb{R}}\frac{(z_t(\alpha)-z_t(\alpha-\beta))^\perp\cdot\partial_\alpha z(\alpha)}{|z(\alpha)-z(\alpha-\beta)|^2}\partial_\alpha\varpi_1(\alpha-\beta)d\beta\\
&+\frac{1}{\pi}PV\int_{\mathbb{R}}\frac{(z_t(\alpha)-z_t(\alpha-\beta))^\perp\cdot\partial_\alpha^2 z(\alpha)}{|z(\alpha)-z(\alpha-\beta)|^2}\varpi_1(\alpha-\beta)d\beta\\
&-\frac{2}{\pi}PV\int_{\mathbb{R}}\frac{(\Delta z_t)^\perp\cdot\partial_\alpha z(\alpha)\Delta z\cdot\Delta\partial_\alpha z}{|z(\alpha)-z(\alpha-\beta)|^4}\varpi_1(\alpha-\beta)d\beta\\
&\equiv S_1^1+S_1^2+S_1^3+S_1^4.\end{aligned}$$

As we could see in the evolution of the arc-chord condition, using the definitions (2.21), (2.22) and (2.23), we can write $\Delta z_t = J_1+J_2+J_3$.

For S_1^1, we split

$$\begin{aligned}S_1^1 &= \frac{1}{\pi}PV\int_{\mathbb{R}}\frac{(\partial_\alpha z_t(\alpha)-\partial_\alpha z_t(\alpha-\beta))^\perp\cdot\partial_\alpha z(\alpha)}{|z(\alpha)-z(\alpha-\beta)|^2}(\varpi_1(\alpha-\beta)-\varpi_1(\alpha))d\beta\\
&+\frac{1}{\pi}PV\int_{\mathbb{R}}\frac{(\partial_\alpha z_t(\alpha)-\partial_\alpha z_t(\alpha-\beta))^\perp\cdot\partial_\alpha z(\alpha)}{|z(\alpha)-z(\alpha-\beta)|^2}\varpi_1(\alpha)d\beta\\
&\equiv S_1^{11}+S_1^{12}.\end{aligned}$$

Since $\varpi_1(\alpha-\beta)-\varpi_1(\alpha)=\beta\int_0^1\partial_\alpha\varpi_1(\alpha-\beta t)dt$ and $\partial_\alpha z_t(\alpha)-\partial_\alpha z_t(\alpha-\beta)=\beta\int_0^1\partial_\alpha^2 z_t(\alpha+\beta-\beta t)dt$ and we have seen (2.25) then we have controlled S_1^{11}.

For S_1^{12}, computing

$$B(\alpha,\beta)=\frac{1}{|\Delta z|^2}-\frac{1}{\beta^2|\partial_\alpha z(\alpha)|^2}=\frac{\beta\int_0^1\int_0^1\partial_\alpha^2 z(\psi)(t-1)dtds\cdot\int_0^1\partial_\alpha z(\alpha)+\partial_\alpha z(\phi)dt}{|\partial_\alpha z(\alpha)|^2|\Delta z|^2}$$

we split

$$\begin{aligned}S_1^{12} &= \frac{1}{\pi}PV\int_{\mathbb{R}}(\beta\int_0^1\partial_\alpha^2 z_t(\alpha-\beta+\beta t)^\perp)dt\cdot\partial_\alpha z(\alpha)B(\alpha,\beta)\varpi_1(\alpha)d\beta\\
&+\frac{1}{\pi}PV\int_{\mathbb{R}}\frac{(\partial_\alpha z_t(\alpha)-\partial_\alpha z_t(\alpha-\beta))^\perp\cdot\partial_\alpha z(\alpha)}{\beta^2|\partial_\alpha z(\alpha)|^2}\varpi_1(\alpha)d\beta\\
&\le \exp C|||z,h|||^2+S_1^{121}.\end{aligned}$$

Here, we have

$$S_1^{121}=\Lambda(\partial_\alpha^\perp z_t)\cdot\partial_\alpha z(\alpha)\frac{\varpi_1(\alpha)}{|\partial_\alpha z(\alpha)|^2},$$

then

$$|S_1^{121}|\le\|\mathcal{F}(z)\|_{L^\infty}\|z\|_{\mathcal{C}^1}\|\varpi_1\|_{L^\infty}|\Lambda(\partial_\alpha z_t)|.$$

Thus,

$$\|S_1^1\|_{L^2}\le\exp C|||z,h|||^2\|\Lambda(\partial_\alpha z_t)\|_{L^2}\le\exp C|||z,h|||^2\|\partial_\alpha^2 z_t\|_{L^2}\le\exp C|||z,h|||^2.$$

For S_1^2 we do the same thing,

$$
\begin{aligned}
S_1^2 &= \frac{1}{\pi}PV\int_{\mathbb{R}}\beta\int_0^1 \partial_\alpha^\perp z_t(\alpha-\beta+\beta t)dt\cdot\partial_\alpha z(\alpha)B(\alpha,\beta)\partial_\alpha\varpi_1(\alpha-\beta)d\beta\\
&+\frac{1}{\pi}PV\int_{\mathbb{R}}\frac{\int_0^1\partial_\alpha^\perp z_t(\alpha-\beta+\beta t)dt\cdot\partial_\alpha z(\alpha)}{\beta|\partial_\alpha z(\alpha)|^2}\partial_\alpha\varpi_1(\alpha-\beta)d\beta\\
&\leq \exp c|||z,h|||^2\\
&+\frac{1}{\pi}PV\int_{\mathbb{R}}\frac{\int_0^1\int_0^1\partial_\alpha^2 z_t(\alpha-\beta s+\beta ts)^\perp(1-t)dsdt\cdot\partial_\alpha z(\alpha)}{|\partial_\alpha z(\alpha)|^2}\partial_\alpha\varpi_1(\alpha-\beta)d\beta\\
&+\frac{\partial_\alpha^\perp z_t(\alpha)\cdot\partial_\alpha z(\alpha)}{|\partial_\alpha z(\alpha)|^2}H(\partial_\alpha\varpi_1).
\end{aligned}
$$

Therefore, using (2.24) and (2.25)

$$\|S_1^2\|_{L^2}\leq \exp C|||z,h|||^2.$$

For S_1^3 exactly the same as in S_1^2. And for S_1^4, computing:

$$
\begin{aligned}
C(\alpha,\beta) &= \frac{1}{|\Delta z|^4}-\frac{1}{\beta^4|\partial_\alpha z(\alpha)|^4}\\
&=\frac{\beta\int_0^1\int_0^1\partial_\alpha^2 z(\psi)(t-1)dsdt\cdot\int_0^1\partial_\alpha z(\alpha)+\partial_\alpha z(\phi)dt\int_0^1|\partial_\alpha z(\alpha)|^2+|\partial_\alpha z(\phi)|^2dt}{|\Delta z|^4|\partial_\alpha z(\alpha)|^4}
\end{aligned}
$$

where $\psi=\alpha-\beta+\beta t+\beta s-\beta ts$ and $\phi=\alpha-\beta+\beta t$. We have,

$$
\begin{aligned}
S_1^4 &= -\frac{2}{\pi}PV\int_{\mathbb{R}}\beta^3\int_0^1\partial_\alpha z_t(\phi)^\perp dt\cdot\partial_\alpha z(\alpha)\int_0^1\partial_\alpha z(\phi)dt\cdot\int_0^1\partial_\alpha^2 z(\phi)dtC(\alpha,\beta)\varpi_1(\alpha-\beta)d\beta\\
&-\frac{2}{\pi}PV\int_{\mathbb{R}}\frac{\int_0^1\partial_\alpha z_t(\phi)^\perp dt\cdot\partial_\alpha z(\alpha)\int_0^1\partial_\alpha z(\phi)dt\cdot\int_0^1\partial_\alpha^2 z(\phi)dt}{\beta|\partial_\alpha z(\alpha)|^4}\varpi_1(\alpha-\beta)d\beta\\
&\leq \exp C|||z,h|||^2+S_1^{41}.
\end{aligned}
$$

It is easy to get

$$S_1^{41}\leq \exp C|||z,h|||^2-2\frac{\partial_\alpha^\perp z_t(\alpha)\cdot\partial_\alpha z(\alpha)\partial_\alpha z(\alpha)\cdot\partial_\alpha^2 z(\alpha)}{|\partial_\alpha z(\alpha)|^4}H(\varpi_1).$$

Then,

$$\|S_1^4\|_{L^2}\leq \exp C|||z,h|||^2.$$

Therefore, $\|\partial_\alpha S_1\|_{L^2}\leq \exp C|||z,h|||^2$. We have controlled $\|\partial_\alpha S_2\|_{L^2}+\|\partial_\alpha S_3\|_{L^2}$ in the same way. Now let us estimate $\|\partial_\alpha R_2\|_{L^2}$. We have,

$$
\begin{aligned}
R_2(\varpi_2) &= \frac{1}{\pi}PV\int_{\mathbb{R}}\frac{z_t^\perp(\alpha)\cdot\partial_\alpha z(\alpha)}{|z(\alpha)-h(\alpha-\beta)|^2}\varpi_2(\alpha-\beta)d\beta\\
&+\frac{1}{\pi}PV\int_{\mathbb{R}}\frac{(z(\alpha)-h(\alpha-\beta))^\perp\cdot\partial_\alpha z_t(\alpha)}{|z(\alpha)-h(\alpha-\beta)|^2}\varpi_2(\alpha-\beta)d\beta\\
&-\frac{2}{\pi}PV\int_{\mathbb{R}}\frac{(\Delta zh)^\perp\cdot\partial_\alpha z(\alpha)\Delta zh\cdot z_t(\alpha)}{|z(\alpha)-h(\alpha-\beta)|^4}\varpi_2(\alpha-\beta)d\beta\\
&\equiv S_4+S_5+S_6.
\end{aligned}
$$

Then,

$$
\begin{aligned}
S_4 &\leq C\|d(z,h)\|_{L^\infty}\|z\|_{\mathcal{C}^1}\|z_t\|_{L^2}\|\varpi_1\|_{L^2},\\
S_5 &\leq C\|d(z,h)\|_{L^\infty}^{\frac{1}{2}}\|z\|_{\mathcal{C}^1}\|\varpi_1\|_{L^2},\\
S_6 &\leq C\|d(z,h)\|_{L^\infty}^{\frac{1}{2}}\|z\|_{\mathcal{C}^1}\|z_t\|_{L^2}\|\varpi_1\|_{L^2}.
\end{aligned}
$$

In conclusion,

$$\|\partial_\alpha R_2(\varpi_2)\|_{L^2} \leq \exp C|||z,h|||^2.$$

Moreover, $\partial_\alpha R_3$ is like $\partial_\alpha R_2$ changing z with h, then $\|\partial_\alpha R_3(\varpi_1)\|_{L^2} \leq \exp C|||z,h|||^2$. Thus, we have controlled $\|\partial_t\varpi_1\|_{L^2}$ and $\|\partial_t\varpi_2\|_{L^2}$. Finally, in order to control $\|\partial_t\varpi_1\|_{\mathcal{C}^\delta}$ we will use

$$\|\partial_t\varpi_1\|_{\mathcal{C}^\delta} \leq C(\|T_1(\partial_t\varpi_1)\|_{\mathcal{C}^\delta} + \|T_2(\partial_t\varpi_2)\|_{\mathcal{C}^\delta} + \|R_1(\varpi_1)\|_{\mathcal{C}^\delta} + \|R_2(\varpi_2)\|_{\mathcal{C}^\delta} + \|\partial_\alpha z_t\|_{\mathcal{C}^\delta}).$$

Using the Lemma 2.1.1,

$$
\begin{aligned}
\|T_1(\varpi_1)\|_{\mathcal{C}^\delta} &\leq \|T_1(\varpi_1)\|_{H^1} \leq C\|\mathcal{F}(z)\|_{L^\infty}^2\|z\|_{\mathcal{C}^{2,\delta}}^4\|\partial_t\varpi_1\|_{L^2},\\
\|T_2(\varpi_2)\|_{\mathcal{C}^\delta} &\leq \|T_2(\varpi_2)\|_{H^1} \leq C\|d(z,h)\|_{L^\infty}^2\|h\|_{\mathcal{C}^{2,\delta}}^4\|\partial_t\varpi_2\|_{L^2},
\end{aligned}
$$

for $\delta \leq \frac{1}{2}$. We have already seen $\|R_1(\varpi_1)\|_{H^1} + \|R_2(\varpi_2)\|_{H^1} \leq \exp C|||z,h|||^2$ then

$$\|R_1(\varpi_1)\|_{\mathcal{C}^\delta} + \|R_2(\varpi_2)\|_{\mathcal{C}^\delta} \leq \exp C|||z,h|||^2.$$

Finally let us observe that $\|\partial_\alpha z_t\|_{\mathcal{C}^\delta} \leq \|z_t\|_{H^2}$ which is controlled by $\|\partial_\alpha^2 z_t\|_{L^2}$. The upper bound

$$|\sigma_t(\alpha,t)| \leq \exp C|||z,h|||^2$$

gives us

$$m'(t) \geq -\exp C|||z,h|||^2$$

for almost every t. And a further integration yields

$$m(t) \geq m(0) - \int_0^t \exp C|||z,h|||^2(s)ds.$$

■

2.9 Regularization and Local-existence

This step is classical, then we only sketch this procedure. We regularize the problem as follows:

$$
\begin{aligned}
&z_t^\varepsilon(\alpha,t) = BR(\varpi_1^\varepsilon, z^\varepsilon)_{z^\varepsilon}(\alpha,t) + BR(\varpi_2^\varepsilon, h^\varepsilon)_{z^\varepsilon}(\alpha,t) + c^\varepsilon(\alpha,t)\partial_\alpha z^\varepsilon(\alpha,t)\\
&z^\varepsilon(\alpha,0) = \phi_\varepsilon * z_0(\alpha)
\end{aligned}
$$

where

$$
\begin{aligned}
c^{\varepsilon}(\alpha,t) &= \frac{\alpha+\pi}{2\pi A^{\varepsilon}(t)}\int_{\mathbb{T}}\partial_\alpha z^{\varepsilon}(\beta,t)\cdot\partial_\alpha(BR(\varpi_1^{\varepsilon},z^{\varepsilon})_{z^{\varepsilon}}+BR(\varpi_2^{\varepsilon},h^{\varepsilon})_{z^{\varepsilon}})d\beta\\
&-\int_{-\pi}^{\alpha}\frac{\partial_\alpha z^{\varepsilon}(\beta,t)}{A^{\varepsilon}(t)}\cdot\partial_\alpha(BR(\varpi_1^{\varepsilon},z^{\varepsilon})_{z^{\varepsilon}}+BR(\varpi_2^{\varepsilon},h^{\varepsilon})_{z^{\varepsilon}})d\beta,\\
\varpi_1^{\varepsilon}(\alpha,t) &= -2\frac{\mu^2-\mu^1}{\mu^2+\mu^1}\phi_\varepsilon*\phi_\varepsilon*(BR(\varpi_1^{\varepsilon},z^{\varepsilon})_{z^{\varepsilon}}+BR(\varpi_2^{\varepsilon},h^{\varepsilon})_{z^{\varepsilon}})\cdot\partial_\alpha z^{\varepsilon}(\alpha,t)\\
&-2\kappa^1\frac{\rho^2-\rho^1}{\mu^2+\mu^1}g\phi_\varepsilon*\phi_\varepsilon*\partial_\alpha z_2^{\varepsilon}(\alpha,t)\\
\varpi_2(\alpha,t) &= -2\frac{\kappa^2-\kappa^1}{\kappa^2+\kappa^1}\phi_\varepsilon*\phi_\varepsilon*(BR(\varpi_1^{\varepsilon},z^{\varepsilon})_{h^{\varepsilon}}+BR(\varpi_2^{\varepsilon},h^{\varepsilon})_{h^{\varepsilon}})\cdot\partial_\alpha h^{\varepsilon}(\alpha)
\end{aligned}
$$

for $\phi\in\mathcal{C}_c^{\infty}$, $\phi(\alpha)\geq 0$, $\phi(-\alpha)=\phi(\alpha)$, $\int_{\mathbb{R}}\phi(\alpha)d\alpha=1$ and $\phi_\varepsilon(\alpha)=\phi(\frac{\alpha}{\varepsilon})/\varepsilon$. Using the same techniques that in the above section, we can prove that:

$$
\begin{aligned}
\frac{d}{dt}\|z^{\varepsilon}\|^2_{H^k}(t) &\leq \exp C(\|\mathcal{F}(z^{\epsilon})\|^2_{L^{\infty}(S)}+\|d(z^{\varepsilon},h)\|_{L^{\infty}}+\|z^{\epsilon}\|^2_{H^k(S)})\\
&-\frac{\kappa^1}{2\pi(\mu^2+\mu^1)}\int_{\mathbb{T}}\frac{\sigma^{\varepsilon}(\alpha,t)}{A^{\varepsilon}(t)}\phi_\varepsilon*\partial_\alpha^k z^{\varepsilon}\cdot\Lambda(\phi_\varepsilon*\partial_\alpha^k z^{\varepsilon})d\alpha
\end{aligned}
$$

where

$$
\sigma^{\varepsilon}(\alpha,t)=\frac{\mu^2-\mu^1}{\kappa^1}(BR(\varpi_1^{\varepsilon},z^{\varepsilon})_{z^{\varepsilon}}+BR(\varpi_2^{\varepsilon},h^{\varepsilon})_{z^{\varepsilon}})\cdot\partial_\alpha^{\perp}z^{\varepsilon}(\alpha,t)+g(\rho^2-\rho^1)\partial_\alpha z^{\varepsilon}(\alpha,t)
$$

The procedure is the same except when we estimate the corresponding term I_1^4. In this regularize case we call ${}^{\varepsilon}I_1^4$ and we will need:

$$
\|\phi_\varepsilon*(gf)-g\phi_\varepsilon(f)\|_{H^1}\leq C\|g\|_{\mathcal{C}^1}\|f\|_{L^2} \tag{2.26}
$$

where C is independent of ε. The most singular terms are:

$$
{}^{\varepsilon}I_1^{42}=\frac{1}{2\pi A^{\varepsilon}(t)}\int_{\mathbb{T}}\Lambda(\partial_\alpha^3 z^{\varepsilon}(\alpha)\cdot\partial_\alpha^{\perp}z^{\varepsilon}(\alpha))\phi_\varepsilon*\phi_\varepsilon*\partial_\alpha^2\varpi_1^{\varepsilon}(\alpha)d\alpha.
$$

Since $\varpi_1^{\varepsilon}=-\lambda_1T_1^{\varepsilon}(\varpi_1^{\varepsilon})-\lambda_1T_2^{\varepsilon}(\varpi_2^{\varepsilon})-N\partial_\alpha z_2^{\varepsilon}(\alpha)$ we split ${}^{\varepsilon}I_1^{42}$,

$$
\begin{aligned}
{}^{\varepsilon}I_1^{421} &= \frac{-N}{2\pi A^{\varepsilon}(t)}\int_{\mathbb{T}}\Lambda(\partial_\alpha^3 z^{\varepsilon}(\alpha)\cdot\partial_\alpha^{\perp}z^{\varepsilon}(\alpha))\phi_\varepsilon*\phi_\varepsilon*\partial_\alpha^3 z_2^{\varepsilon}(\alpha)d\alpha,\\
{}^{\varepsilon}I_1^{422} &= \frac{-\lambda_1}{2\pi A^{\varepsilon}(t)}\int_{\mathbb{T}}\Lambda(\partial_\alpha^3 z^{\varepsilon}\cdot\partial_\alpha^{\perp}z^{\varepsilon})(\alpha)\phi_\varepsilon*\phi_\varepsilon*(\partial_\alpha^2T_1^{\varepsilon}(\varpi_1^{\varepsilon})+\partial_\alpha^2T_2^{\varepsilon}(\varpi_2^{\varepsilon}))d\alpha.
\end{aligned}
$$

We can write ${}^{\varepsilon}I_1^{421}=L_1^{\varepsilon}+L_2^{\varepsilon}$ where

$$
\begin{aligned}
{}^{\varepsilon}L_1 &= \frac{N}{2\pi A^{\varepsilon}(t)}\int_{\mathbb{T}}\Lambda(\phi_\varepsilon*\partial_\alpha^3 z_1^{\varepsilon}\partial_\alpha z_2^{\varepsilon})(\alpha)\phi_\varepsilon*\partial_\alpha^3 z_2^{\varepsilon}(\alpha)d\alpha,\\
{}^{\varepsilon}L_2 &= \frac{-N}{2\pi A^{\varepsilon}(t)}\int_{\mathbb{T}}\Lambda(\phi_\varepsilon*\partial_\alpha^3 z_2^{\varepsilon}\partial_\alpha z_1^{\varepsilon})(\alpha)\phi_\varepsilon*\partial_\alpha^3 z_2^{\varepsilon}(\alpha)d\alpha,
\end{aligned}
$$

We split $L_1^\varepsilon = M_1^\varepsilon + M_2^\varepsilon + M_3^\varepsilon + M_4^\varepsilon$ where,

$$
\begin{aligned}
M_1^\varepsilon &= \frac{N}{2\pi A^\varepsilon(t)}\int_{\mathbb{T}} \Lambda(\phi_\varepsilon * (\partial_\alpha^3 z_1^\varepsilon \partial_\alpha z_2^\varepsilon) - \partial_\alpha z_2^\varepsilon \phi_\varepsilon * (\partial_\alpha^3 z_1^\varepsilon))(\alpha)\phi_\varepsilon * \partial_\alpha^3 z_2^\varepsilon(\alpha)d\alpha,\\
M_2^\varepsilon &= \frac{N}{2\pi A^\varepsilon(t)}\int_{\mathbb{T}} (\Lambda(\partial_\alpha z_2^\varepsilon \phi_\varepsilon * (\partial_\alpha^3 z_1^\varepsilon))(\alpha) - \partial_\alpha z_2^\varepsilon \Lambda(\phi_\varepsilon * \partial_\alpha^3 z_1^\varepsilon)(\alpha))\phi_\varepsilon * \partial_\alpha^3 z_2^\varepsilon(\alpha)d\alpha,\\
M_3^\varepsilon &= \frac{N}{2\pi A^\varepsilon(t)}\int_{\mathbb{T}} \Lambda(\phi_\varepsilon * \partial_\alpha^3 z_1^\varepsilon)(\alpha)(\partial_\alpha z_2^\varepsilon(\alpha)\phi_\varepsilon * \partial_\alpha^3 z_2^\varepsilon(\alpha) - \phi_\varepsilon * (\partial_\alpha z_2^\varepsilon \partial_\alpha^3 z_2^\varepsilon)(\alpha))d\alpha,\\
M_4^\varepsilon &= \frac{N}{2\pi A^\varepsilon(t)}\int_{\mathbb{T}} \Lambda(\phi_\varepsilon * \partial_\alpha^3 z_1^\varepsilon)(\alpha)\phi_\varepsilon * (\partial_\alpha z_2^\varepsilon \partial_\alpha^3 z_2^\varepsilon)(\alpha)d\alpha.
\end{aligned}
$$

Thus, using (2.26)

$$
\begin{aligned}
M_1^\varepsilon &\le C\|\mathcal{F}(z^\varepsilon)\|_{L^\infty}\|\Lambda(\phi_\varepsilon * (\partial_\alpha^3 z_1^\varepsilon \partial_\alpha z_2^\varepsilon) - \partial_\alpha z_2^\varepsilon \phi_\varepsilon * (\partial_\alpha^3 z_1^\varepsilon))\|_{L^2}\|\phi_\varepsilon * \partial_\alpha^3 z_2^\varepsilon\|_{L^2}\\
&\le C\|\mathcal{F}(z^\varepsilon)\|_{L^\infty}\|\phi_\varepsilon * (\partial_\alpha^3 z_1^\varepsilon \partial_\alpha z_2^\varepsilon) - \partial_\alpha z_2^\varepsilon \phi_\varepsilon * (\partial_\alpha^3 z_1^\varepsilon)\|_{H^1}\|\partial_\alpha^3 z_2^\varepsilon\|_{L^2}\\
&\le C\|\mathcal{F}(z^\varepsilon)\|_{L^\infty}\|\partial_\alpha z_2^\varepsilon\|_{\mathcal{C}^1}\|\partial_\alpha^3 z_1^\varepsilon\|_{L^2}\|\partial_\alpha^3 z_2^\varepsilon\|_{L^2}.
\end{aligned}
$$

Using the commutator estimations for the Λ operator,

$$
M_2^\varepsilon \le C\|\mathcal{F}(z^\varepsilon)\|_{L^\infty}\|\partial_\alpha z_2^\varepsilon\|_{\mathcal{C}^{1,\delta}}\|\phi_\varepsilon * (\partial_\alpha^3 z_1^\varepsilon)\|_{L^2}\|\phi_\varepsilon * (\partial_\alpha^3 z_2^\varepsilon)\|_{L^2},
$$

then we have M_2^ε controlled. We can write,

$$
M_3^\varepsilon = \frac{N}{2\pi A^\varepsilon(t)}\int_{\mathbb{T}} \phi_\varepsilon * \partial_\alpha^3 z_1^\varepsilon(\alpha)\Lambda(\partial_\alpha z_2^\varepsilon \phi_\varepsilon * \partial_\alpha^3 z_2^\varepsilon - \phi_\varepsilon * (\partial_\alpha z_2^\varepsilon \partial_\alpha^3 z_2^\varepsilon))(\alpha)d\alpha,
$$

thus M_3^ε is estimated like M_1^ε. Using that $\partial_\alpha z_2^\varepsilon \partial_\alpha^3 z_2^\varepsilon = -\partial_\alpha z_1^\varepsilon \partial_\alpha^3 z_1^\varepsilon - |\partial_\alpha^2 z^\varepsilon|^2$, we can separate $M_4^\varepsilon = N_1^\varepsilon + N_2^\varepsilon$ where:

$$
\begin{aligned}
N_1^\varepsilon &= -\frac{N}{2\pi A^\varepsilon(t)}\int_{\mathbb{T}} \Lambda(\phi_\varepsilon * \partial_\alpha^3 z_1^\varepsilon)(\alpha)\phi_\varepsilon * (|\partial_\alpha^2 z^\varepsilon|^2)(\alpha)d\alpha,\\
N_2^\varepsilon &= -\frac{N}{2\pi A^\varepsilon(t)}\int_{\mathbb{T}} \Lambda(\phi_\varepsilon * \partial_\alpha^3 z_1^\varepsilon)(\alpha)\phi_\varepsilon * (\partial_\alpha z_1^\varepsilon \partial_\alpha^3 z_1^\varepsilon)(\alpha)d\alpha.
\end{aligned}
$$

Integrating by parts,

$$
N_1^\varepsilon \le C\|\mathcal{F}(z^\varepsilon)\|_{L^\infty}\|\phi_\varepsilon * \partial_\alpha^3 z_1^\varepsilon\|_{L^2}\|\phi_\varepsilon * \partial_\alpha^2 z^\varepsilon\|_{H^1}.
$$

Therefore,

$$
\begin{aligned}
L_1^\varepsilon \le{}& \exp C(\|\mathcal{F}(z^\epsilon)\|_{L^\infty(S)}^2 + \|d(z^\varepsilon,h)\|_{L^\infty} + \|z^\epsilon\|_{H^3(S)}^2)\\
&- \frac{N}{2\pi A^\varepsilon(t)}\int_{\mathbb{T}} \partial_\alpha z_1^\varepsilon(\alpha)\phi_\varepsilon * \partial_\alpha^3 z_1^\varepsilon(\alpha)\Lambda(\phi_\varepsilon * \partial_\alpha^3 z_1^\varepsilon)(\alpha)d\alpha.
\end{aligned}
$$

If we add L_2^ε,

$$
\begin{aligned}
{}^\varepsilon I_1^{421} \le{}& \exp C(\|\mathcal{F}(z^\epsilon)\|_{L^\infty(S)}^2 + \|d(z^\varepsilon,h)\|_{L^\infty} + \|z^\epsilon\|_{H^3(S)}^2)\\
&- \frac{N}{2\pi A^\varepsilon(t)}\int_{\mathbb{T}} \partial_\alpha z_1^\varepsilon(\alpha)\phi_\varepsilon * \partial_\alpha^3 z^\varepsilon(\alpha)\cdot\Lambda(\phi_\varepsilon * \partial_\alpha^3 z^\varepsilon)(\alpha)d\alpha.
\end{aligned}
$$

In the same way for ${}^{\varepsilon}I_1^{422}$, we will find the other part of $\sigma^{\varepsilon}(\alpha,t)$ and we get the desire estimate. The next step is to integrate during a time T independent of ε. Let us observe that if $\phi_{\varepsilon} * z_0(\alpha) \in H^k$, then we have the solution $z^{\varepsilon} \in \mathcal{C}^1([0,T^{\varepsilon}],H^k)$. If $\sigma(\alpha,0)>0$, there exists T^{ε} dependent of ε where $\sigma^{\varepsilon}(\alpha,t)>0$. Then for $t \leq T^{\varepsilon}$ with our a priori estimates and the fact that

$$f(\alpha)\Lambda f(\alpha) - \frac{1}{2}\Lambda(f^2)(\alpha) \geq 0$$

we get

$$\begin{aligned}
&\frac{d}{dt}\|z^{\varepsilon}\|^2_{H^k}(t) \leq \exp C(\|\mathcal{F}(z^{\epsilon})\|^2_{L^\infty(S)} + \|d(z^{\varepsilon},h)\|_{L^\infty} + \|z^{\epsilon}\|^2_{H^k(S)}) \\
&- \frac{\kappa^1}{4\pi(\mu^2+\mu^1)A^{\varepsilon}(t)}\int_{\mathbb{T}} \sigma^{\varepsilon}(\alpha,t)\Lambda(|\phi_{\varepsilon} * \partial_\alpha^k z^{\varepsilon}|^2)(\alpha)d\alpha.
\end{aligned}$$

Since

$$\begin{aligned}
&\|\Lambda\sigma^{\varepsilon}\|_{L^\infty} \leq C\|\sigma^{\varepsilon}\|_{H^2} \leq C(\|BR(\varpi_1^{\varepsilon},z^{\varepsilon})_{z^{\varepsilon}} + BR(\varpi_2^{\varepsilon},h^{\varepsilon})_{z^{\varepsilon}}\|_{L^2} \\
&+ \|\partial_\alpha^2 BR(\varpi_1^{\varepsilon},z^{\varepsilon})_{z^{\varepsilon}} + \partial_\alpha^2 BR(\varpi_2^{\varepsilon},h^{\varepsilon})_{z^{\varepsilon}}\|_{L^2} + 1)\|z^{\varepsilon}\|_{H^3},
\end{aligned}$$

then

$$\frac{d}{dt}\|z^{\varepsilon}\|^2_{H^k}(t) \leq \exp C(\|\mathcal{F}(z^{\epsilon})\|^2_{L^\infty(S)} + \|d(z^{\varepsilon},h)\|_{L^\infty} + \|z^{\epsilon}\|^2_{H^k(S)}).$$

We have seen in sections 2.6 and 2.7 that

$$\frac{d}{dt}\|\mathcal{F}(z^{\varepsilon})\|^2_{L^\infty} \leq \exp C(\|\mathcal{F}(z^{\epsilon})\|^2_{L^\infty(S)} + \|d(z^{\varepsilon},h)\|_{L^\infty} + \|z^{\epsilon}\|^2_{H^k(S)})$$

and

$$\frac{d}{dt}\|d(z^{\varepsilon},h^{\varepsilon})\|^2_{L^\infty} \leq \exp C(\|\mathcal{F}(z^{\epsilon})\|^2_{L^\infty(S)} + \|d(z^{\varepsilon},h)\|_{L^\infty} + \|z^{\epsilon}\|^2_{H^k(S)}).$$

Therefore,

$$\frac{d}{dt}(\|\mathcal{F}(z^{\varepsilon})\|^2_{L^\infty} + \|d(z^{\varepsilon},h^{\varepsilon})\|^2_{L^\infty} + \|z^{\varepsilon}\|^2_{H^k}) \leq \exp C(\|\mathcal{F}(z^{\epsilon})\|^2_{L^\infty(S)} + \|d(z^{\varepsilon},h)\|_{L^\infty} + \|z^{\epsilon}\|^2_{H^k(S)}).$$

Integrating,

$$\begin{aligned}
&\|\mathcal{F}(z^{\varepsilon})\|^2_{L^\infty} + \|d(z^{\varepsilon},h^{\varepsilon})\|^2_{L^\infty} + \|z^{\varepsilon}\|^2_{H^k} \\
&\leq -\frac{1}{C}\ln(-t + \exp(-C(\|\mathcal{F}(z_0)\|^2_{L^\infty} + \|d(z_0,h)\|^2_{L^\infty} + \|z_0\|^2_{H^k}))).
\end{aligned}$$

Since $m^{\varepsilon}(t) \geq m(0) - \int_0^t \exp C(\|\mathcal{F}(z^{\epsilon})\|^2_{L^\infty(S)} + \|d(z^{\varepsilon},h)\|_{L^\infty} + \|z^{\epsilon}\|^2_{H^k(S)})(s)ds$, where $m^{\varepsilon}(t) = \min_{\alpha\in\mathbb{T}} \sigma^{\varepsilon}(\alpha,t)$ for $t \leq T^{\varepsilon}$, using the above estimations

$$\begin{aligned}
&m^{\epsilon}(t) \geq m(0) + C(\|\mathcal{F}(z_0)\|^2_{L^\infty} + \|d(z_0,h)\|^2_{L^\infty} + \|z_0\|^2_{H^k}) \\
&+ \ln(-t + \exp(-C(\|\mathcal{F}(z_0)\|^2_{L^\infty} + \|d(z_0,h)\|^2_{L^\infty} + \|z_0\|^2_{H^k})))
\end{aligned}$$

for $t \leq T^{\varepsilon}$. Now if we $\varepsilon \to 0$ we have $T^{\varepsilon} \nrightarrow 0$. This is because if we take $T = \min(T^1, T^2)$ where T^1 satisfies,

$$\begin{aligned} &m(0) + C(\|\mathcal{F}(z_0)\|^2_{L^\infty} + \|d(z_0,h)\|^2_{L^\infty} + \|z_0\|^2_{H^k}) \\ &+ \ln(-T^1 + \exp(-C(\|\mathcal{F}(z_0)\|^2_{L^\infty} + \|d(z_0,h)\|^2_{L^\infty} + \|z_0\|^2_{H^k}))) > 0 \end{aligned}$$

and T_2 satisfies,

$$-\frac{1}{C}\ln(-T^2 + \exp(-C(\|\mathcal{F}(z_0)\|^2_{L^\infty} + \|d(z_0,h)\|^2_{L^\infty} + \|z_0\|^2_{H^k}))) < \infty.$$

For $t \leq T$ we have $m^{\varepsilon}(t) > 0$ and

$$\begin{aligned} &\|\mathcal{F}(z^{\varepsilon})\|^2_{L^\infty} + \|d(z^{\varepsilon},h)\|^2_{L^\infty} + \|z^{\varepsilon}\|^2_{H^k} \\ &\leq -\frac{1}{C}\ln(-T^2 + \exp(-C(\|\mathcal{F}(z_0)\|^2_{L^\infty} + \|d(z_0,h)\|^2_{L^\infty} + \|z_0\|^2_{H^k}))) < \infty \end{aligned}$$

and T only depend on z_0. Then, we have local existence when $\varepsilon \to 0$.

Chapter 3

Finite-time singularities for the one-phase inhomogeneous Muskat Problem

In this chapter we study finite time singularities formation for the interface of one fluid in a porous media with two different permeabilities. We prove that the smoothness of the interface breaks down in finite time into a splash singularity but this is not going to happen into a splat singularity.

In this case, we consider the inhomogeneous one-phase Muskat problem. Therefore the equations which describes our problem are:

$$OIMP = \begin{cases} z_t(\alpha,t) = BR(\varpi_1,z)_z(\alpha,t) + BR(\varpi_2,h)_z(\alpha,t) + c(\alpha,t)\partial_\alpha z(\alpha,t) \\ c(\alpha,t) = \frac{\alpha+\pi}{2\pi A(t)}\int_{\mathbb{T}} \partial_\alpha z(\beta,t)\cdot\partial_\alpha(BR(\varpi_1,z)_z + BR(\varpi_2,h)_z)d\beta \\ -\int_{-\pi}^{\alpha}\frac{\partial_\alpha z(\beta,t)}{A(t)}\cdot\partial_\alpha(BR(\varpi_1,z)_z + BR(\varpi_2,h)_z)d\beta \\ \varpi_1(\alpha,t) = -2(BR(\varpi_1,z)_z + BR(\varpi_2,h)_z)\cdot\partial_\alpha z(\alpha,t) - 2\kappa^1\frac{\rho^2}{\mu^2}g\partial_\alpha z_2(\alpha,t) \\ \varpi_2(\alpha,t) = -2\frac{\kappa^1-\kappa^2}{\kappa^2+\kappa^1}(BR(\varpi_1,z)_h + BR(\varpi_2,h)_h)\cdot\partial_\alpha h(\alpha) \end{cases}$$

where c is taken in such a way that we can assure that $|\partial_\alpha z(\alpha)|^2 \equiv A(t)$, $p(z(\alpha,t),t) = 0$ and the Birkhoff-Rott integrals are given by:

$$BR(\varpi_1,z)_x = \frac{1}{2\pi}PV\int_{\mathbb{R}}\frac{(x-z(\beta,t))^\perp}{|x-z(\beta,t)|^2}\varpi_1(\beta,t)d\beta,$$
$$BR(\varpi_2,h)_x = \frac{1}{2\pi}PV\int_{\mathbb{R}}\frac{(x-h(\beta))^\perp}{|x-h(\beta)|^2}\varpi_2(\beta,t)d\beta.$$

The Rayleigh-Taylor condition can be written by

$$\sigma(\alpha,t) = \frac{\mu^2}{\kappa^1}(BR(\varpi_1,z)_z + BR(\varpi_2,h)_z)\cdot\partial_\alpha^\perp z(\alpha) + \rho^2 g\partial_\alpha z_1(\alpha) > 0. \tag{3.1}$$

Remark 3.0.1. *In particular, Rayleigh-Taylor condition for one-phase inhomogeneous Muskat problem holds under the conditions $\kappa^1 > \kappa^2$ and $h(\alpha)$ being a graph.*

Lemma 3.0.1. *The Rayleigh-Taylor condition holds for $\kappa^1 > \kappa^2$ and $h(\alpha)$ being a graph for the* *one-phase inhomogeneous Muskat problem.*

Proof. Recall that the Rayleigh-Taylor condition is

$$\sigma(\alpha, t) = -\nabla_n p^2(z(\alpha, t)), t) > 0.$$

where ∇_n is the gradient in the direction of n. When we approach to z, n will be $\partial_\alpha^\perp z(\alpha, t)$ and when we approach to h, n equals to $\partial_\alpha^\perp h(\alpha)$.

If we have that the minimum of p^2 in Ω^2 is attained on the free boundary $z(\alpha, t)$ and $\Delta p^2 = 0$ in Ω^2, using Hopf lemma we could conclude that $\nabla_n p^2(z(\alpha, t), t) < 0$ and the Rayleigh-Taylor condition holds.

Then, let us suppose that the minimum of p^2 does not reach on $z(\alpha, t)$. Since p^2 is a harmonic function in Ω^2, the minimum would be on $h(\alpha)$. Therefore, Hopf lemma allows us to get $\nabla_n p^2(h(\alpha), t) > 0$.

Since we consider velocities with mean zero vorticity, we have $u \in L^2(\Omega^3)$ and finite energy settings. Then,

$$\lim_{x_2 \to -\infty} u(x, t) = 0.$$

Darcy's law give us

$$\lim_{x_2 \to -\infty} \partial_{x_1} p^3(x, t) = 0,$$
$$\lim_{x_2 \to -\infty} \partial_{x_2} p^3(x, t) = g\rho^2.$$

Therefore, $p^3 = -g\rho^2 x_2 + o(x_2)$ when $x_2 \to -\infty$. In conclusion, $p^3(x, t) \to +\infty$ when $x_2 \to -\infty$ and the minimum of p^3 in Ω^3 is attained on the boundary $h(\alpha)$. Therefore, using Hopf's lemma, $\nabla_n p^3(h(\alpha), t) < 0$.

Now if $n = (n_1, n_2) = \partial_\alpha^\perp h(\alpha)$, using Darcy's law it is easy to get:

$$\kappa^2(\nabla_n p^3(h(\alpha), t) + g\rho^2 n_2) = \kappa^1(\nabla_n p^2(h(\alpha), t) + g\rho^2 n_2).$$

Therefore,

$$\nabla_n p^2(h(\alpha), t) = \frac{\kappa^2}{\kappa^1}\nabla_n p^3(h(\alpha), t) + \frac{\kappa^2 - \kappa^1}{\kappa^1} g\rho^2 n_2.$$

Looking at the above equation, if $n_2 \geq 0$ and $\kappa^1 > \kappa^2$ we have $\nabla_n p^2(h(\alpha), t) < 0$ which is a contradiction.

In conclusion, if $h(\alpha)$ is a graph and the medium is less permeable at the bottom, we can guarantee $\sigma(\alpha, t) > 0$. ∎

3.1 Non-splat singularity for the one-phase inhomogenous Muskat Problem

3.1.1 Instant analyticity

Here we show the main estimates that provide instant analyticity into the strip $S(t) = \{\alpha + i\zeta : |\zeta| < \lambda t\}$ for each t. To do that we will need the estimates (2.16) and (2.20) from sections 2.2 and

2.3: Let $\varpi = (\varpi_1, \varpi_2)$,

$$\|\varpi\|_{H^k} \leq \exp C(\|\mathcal{F}(z)\|^2_{L^\infty} + \|d(z,h)\|^2_{L^\infty} + \|z\|^2_{H^{k+1}}),$$

for $k \geq 2$.

$$\|BR(\varpi_i, u)_v\|_{H^k} \leq \exp C(\|\mathcal{F}(z)\|^2_{L^\infty} + \|d(z,h)\|^2_{L^\infty} + \|z\|^2_{H^{k+1}}),$$

for $i = 1, 2$, $u, v = z, h$ and $k \geq 2$.

These estimates follows also into the complex strip S, since the time derivative plays no role and hence any extra term appears in relation with the terms in Chapter 2.

Theorem 3.1.1. *Let $z(\alpha, 0) = z_0(\alpha) \in H^4$,$h(\alpha) \in H^4$, $\mathcal{F}(z_0)(\alpha,\beta) \in L^\infty$, $d(z_0, h) \in L^\infty$, $\mathcal{F}(h) \in L^\infty$ and $\sigma(\alpha, 0) > 0$. Then there exists a solution of the Muskat problem $(OIMP)$ $z(\alpha, t)$ defined for $0 < t \leq T$ that continues analytically into the strip $S(t) = \{\alpha \pm i\varsigma : |\varsigma| < \lambda t\}$ for each t. Here, λ and T are determined by upper bounds of the H^4 norm, the initial distance between z and h and the arc-chord constant of the initial data and a positive lower bound of the $\sigma(\alpha, 0)$. Moreover, for $0 < t \leq T$, the quantity*

$$\sum_{\pm} \int (|z(\alpha \pm i\lambda t) - (\alpha \pm i\lambda t)|^2 + |\partial^4_\alpha z(\alpha \pm i\lambda t)|^2) d\alpha$$

is bounded by a constant determinate by upper bounds for the H^4 norm and the arc-chord constant of the initial data and a positive lower bound of $\sigma(\alpha, 0)$ and $d(z_0, h)$.

Proof. We consider the norms:

$$\|z\|^2_{L^2(S)}(t) = \sum_{\pm} \int |z(\alpha \pm i\lambda t, t) - (\alpha \pm i\lambda t, 0)|^2 d\alpha,$$

$$\|z\|^2_{H^k(S)}(t) = \|z\|^2_{L^2(S)}(t) + \sum_{\pm} \int |\partial^k_\alpha z(\alpha \pm i\lambda t, t)|^2 d\alpha.$$

Remark 3.1.1. *Above $|\cdot|$ is the modulus of a vector in $\mathbb{C}^2$.*

For the terms in which only the curve z appears, we proceed as in section 1.1. So we only have to take care of the terms with some h.

It is easy to find that

$$\frac{1}{2}\frac{d}{dt} \int |z(\alpha \pm i\lambda t) - (\alpha \pm i\lambda t, 0)|^2 d\alpha \leq \exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)} + \|d(z,h)\|^2_{L^\infty(S)} + \|z\|^2_{H^3(S)}).$$

In order to simplify the exposition we write $z(\alpha, t) = z(\alpha)$ for a fixed t, and we denote $\alpha \pm i\lambda t \equiv \gamma$.

Next, we check that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt} \int |\partial^4_\alpha z(\gamma)|^2 d\alpha &= \Re \int \overline{\partial^4_\alpha z(\gamma)} \cdot \partial^4_\alpha z_t(\gamma) d\alpha \pm \Re \int \overline{\partial^4_\alpha z(\gamma)} \cdot i\lambda \partial^5_\alpha z(\gamma) d\alpha \\ &\equiv I_1 + I_2.\end{aligned}$$

In the same way as in Chapter 1,

$$I_2 \leq 2\lambda \|\Lambda^{\frac{1}{2}} \partial^4_\alpha z\|^2_{L^2(S)}$$

Since $z_t(\gamma) = BR(\varpi_1, z)_z(\gamma) + BR(\varpi_2, h)_z(\gamma) + c(\gamma)\partial_\alpha z(\gamma)$, we have

$$\begin{aligned} I_1 &= \Re \int \overline{\partial_\alpha^4 z(\gamma)} \cdot \partial_\alpha^4 BR(\varpi_1, z)_z(\gamma) d\alpha + \Re \int \overline{\partial_\alpha^4 z(\gamma)} \cdot \partial_\alpha^4 BR(\varpi_2, h)_z(\gamma) d\alpha \\ &+ \Re \int \overline{\partial_\alpha^4 z(\gamma)} \cdot \partial_\alpha^4 (c(\gamma) \cdot \partial_\alpha z(\gamma)) d\alpha \equiv J_1 + J_2 + J_3. \end{aligned}$$

We decompose $J_1 = I_3 + I_4 + I_5 + I_6 + I_7$, where:

$$\begin{aligned} I_3 &= \frac{1}{2\pi} \Re \int_{\mathbb{T}} \int_{\mathbb{R}} \overline{\partial_\alpha^4 z(\gamma)} \cdot \partial_\alpha^4 \Big(\frac{(z(\gamma) - z(\gamma - \beta))^\perp}{|z(\gamma) - z(\gamma - \beta)|^2} \Big) \varpi_1(\gamma - \beta) d\alpha d\beta, \\ I_4 &= \frac{2}{\pi} \Re \int_{\mathbb{T}} \int_{\mathbb{R}} \overline{\partial_\alpha^4 z(\gamma)} \cdot \partial_\alpha^3 \Big(\frac{(z(\gamma) - z(\gamma - \beta))^\perp}{|z(\gamma) - z(\gamma - \beta)|^2} \Big) \partial_\alpha \varpi_1(\gamma - \beta) d\alpha d\beta, \\ I_5 &= \frac{3}{\pi} \Re \int_{\mathbb{T}} \int_{\mathbb{R}} \overline{\partial_\alpha^4 z(\gamma)} \cdot \partial_\alpha^2 \Big(\frac{(z(\gamma) - z(\gamma - \beta))^\perp}{|z(\gamma) - z(\gamma - \beta)|^2} \Big) \partial_\alpha^2 \varpi_1(\gamma - \beta) d\alpha d\beta, \\ I_6 &= \frac{2}{\pi} \Re \int_{\mathbb{T}} \int_{\mathbb{R}} \overline{\partial_\alpha^4 z(\gamma)} \cdot \partial_\alpha \Big(\frac{(z(\gamma) - z(\gamma - \beta))^\perp}{|z(\gamma) - z(\gamma - \beta)|^2} \Big) \partial_\alpha^3 \varpi_1(\gamma - \beta) d\alpha d\beta, \\ I_7 &= \frac{1}{2\pi} \Re \int_{\mathbb{T}} \int_{\mathbb{R}} \overline{\partial_\alpha^4 z(\gamma)} \cdot \frac{(z(\gamma) - z(\gamma - \beta))^\perp}{|z(\gamma) - z(\gamma - \beta)|^2} \partial_\alpha^4 \varpi_1(\gamma - \beta) d\alpha d\beta. \end{aligned}$$

The terms

$$\begin{aligned} &I_3 + I_4 + I_5 + I_6 \le \exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)} + \|d(z,h)\|^2_{L^\infty(S)} + \|z\|^2_{H^4(S)}) \\ &+ C\|\Im(\frac{\varpi_1}{A(t)})\|_{H^2(S)} \|\Lambda^{\frac{1}{2}} \partial_\alpha^4 z\|_{L^2(S)} \end{aligned}$$

in the same way as in Chapter 1. We need to control the term I_7 because we are going to find the R-T condition. Recall that in our case this conditions is given by (3.1).

With the techniques used before, we will get

$$I_7 \le \exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)} + \|d(z,h)\|^2_{L^\infty(S)} + \|z\|^2_{H^4(S)}) + K_1$$

where

$$K_1 = \frac{1}{2} \Re \int_{\mathbb{T}} \overline{\partial_\alpha^4 z(\gamma)} \cdot \frac{\partial_\alpha^\perp z(\gamma)}{A(t)} H(\partial_\alpha^4 \varpi_1)(\gamma) d\alpha = \frac{1}{2} \Re \int_{\mathbb{T}} \frac{\Lambda(\overline{\partial_\alpha^4 z} \cdot \partial_\alpha^\perp z)(\gamma)}{A(t)} \partial_\alpha^3 \varpi_1(\gamma) d\alpha.$$

Since $\varpi_1(\gamma) = -T_1(\varpi_1)(\gamma) - T_2(\varpi_2)(\gamma) - 2g\kappa^1 \frac{\rho^2}{\mu^2} \partial_\alpha z_2(\gamma)$, we decompose $K_1 = P_1 + P_2$ being

$$\begin{aligned} P_1 &= -g\kappa^1 \frac{\rho^2}{\mu^2} \Re \int_{\mathbb{T}} \frac{\Lambda(\overline{\partial_\alpha^4 z} \cdot \partial_\alpha^\perp z)(\gamma)}{A(t)} \partial_\alpha^4 z_2(\gamma) d\alpha, \\ P_2 &= -\frac{1}{2} \Re \int_{\mathbb{T}} \frac{\Lambda(\overline{\partial_\alpha^4 z} \cdot \partial_\alpha^\perp z)(\gamma)}{A(t)} (\partial_\alpha^3 T_1(\varpi_1)(\gamma) + \partial_\alpha^3 T_2(\varpi_2)(\gamma)) d\alpha. \end{aligned}$$

For P_1 we decompose further $P_1 = Q_1 + Q_2$ where

$$\begin{aligned} Q_1 &= g\kappa \frac{\rho^2}{\mu^2} \Re \int_{\mathbb{T}} \frac{\Lambda(\overline{\partial_\alpha^4 z_1} \partial_\alpha z_2)(\gamma)}{A(t)} \partial_\alpha^4 z_2(\gamma) d\alpha, \\ Q_2 &= -g\kappa \frac{\rho^2}{\mu^2} \Re \int_{\mathbb{T}} \frac{\Lambda(\overline{\partial_\alpha^4 z_2} \partial_\alpha z_1)(\gamma)}{A(t)} \partial_\alpha^4 z_2(\gamma) d\alpha. \end{aligned}$$

Then using commutator estimates we have:

$$
\begin{aligned}
Q_1 \leq &\exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)} + \|d(z,h)\|^2_{L^\infty(S)} + \|z\|^2_{H^4(S)}) \\
&+ g\kappa\frac{\rho^2}{\mu^2}\Re\int_{\mathbb{T}} \frac{\partial_\alpha z_2(\gamma)\Lambda(\overline{\partial^4_\alpha z_1})(\gamma)}{A(t)}\partial^4_\alpha z_2(\gamma)d\alpha.
\end{aligned} \tag{3.2}
$$

The identity

$$
\partial_\alpha z_2(\gamma)\partial^4_\alpha z_2(\gamma) = \partial_\alpha z(\gamma)\cdot\partial^4_\alpha z(\gamma) - \partial_\alpha z_1(\gamma)\partial^4_\alpha z_1(\gamma)
$$

let us write the last term on the right side of (3.2) as the sum of S_1 and S_2 where

$$
\begin{aligned}
S_1 &= g\kappa\frac{\rho^2}{\mu^2}\Re\int_{\mathbb{T}} \frac{\Lambda(\overline{\partial^4_\alpha z_1})(\gamma)}{A(t)}\partial_\alpha z(\gamma)\cdot\partial^4_\alpha z(\gamma)d\alpha, \\
S_2 &= -g\kappa\frac{\rho^2}{\mu^2}\Re\int_{\mathbb{T}} \frac{\Lambda(\overline{\partial^4_\alpha z_1})(\gamma)}{A(t)}\partial_\alpha z_1(\gamma)\partial^4_\alpha z_1(\gamma)d\alpha.
\end{aligned}
$$

For S_1 we use the fact that

$$
\partial_\alpha z(\gamma)\cdot\partial^4_\alpha z(\gamma) = -3\partial^2_\alpha z(\gamma)\cdot\partial^3_\alpha z(\gamma)
$$

and an integration by parts, we get

$$
\begin{aligned}
S_1 &= 3g\kappa\frac{\rho^2}{\mu^2}\Re\int_{\mathbb{T}} \frac{H(\overline{\partial^4_\alpha z_1})(\gamma)}{A(t)}\partial^3_\alpha z(\gamma)\cdot\partial^3_\alpha z(\gamma)d\alpha \\
&+ 3g\kappa\frac{\rho^2}{\mu^2}\Re\int_{\mathbb{T}} \frac{H(\overline{\partial^4_\alpha z_1})(\gamma)}{A(t)}\partial^2_\alpha z(\gamma)\cdot\partial^4_\alpha z(\gamma)d\alpha \\
&\leq \exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)} + \|z\|^2_{H^4(S)}).
\end{aligned}
$$

Again the commutator estimates in Q_2 give us:

$$
\begin{aligned}
Q_2 \leq &\exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)} + \|d(z,h)\|^2_{L^\infty(S)} + \|z\|^2_{H^4(S)}) \\
&- g\kappa\frac{\rho^2}{\mu^2}\Re\int_{\mathbb{T}} \frac{\partial_\alpha z_1(\gamma)}{A(t)}\Lambda(\overline{\partial^4_\alpha z_2})(\gamma)\partial^4_\alpha z_2(\gamma)d\alpha.
\end{aligned}
$$

Therefore,

$$
P_1 \leq \exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)} + \|z\|^2_{H^4(S)}) - g\kappa\frac{\rho^2}{\mu^2}\Re\int_{\mathbb{T}} \frac{\partial_\alpha z_1(\gamma)}{A(t)}\partial^4_\alpha z(\gamma)\cdot\Lambda(\overline{\partial^4_\alpha z})(\gamma)d\alpha.
$$

By (2.20) we can check easily that:

$$
\begin{aligned}
P_2 \leq &\exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)} + \|d(z,h)\|^2_{L^\infty(S)} + \|z\|^2_{H^4(S)}) \\
&- \Re\int_{\mathbb{T}} \frac{\Lambda(\overline{\partial^4_\alpha z}\cdot\partial^\perp_\alpha z)(\gamma)}{A(t)}(BR(\varpi_1,z)_z + BR(\varpi_2,h)_z)\cdot\partial^4_\alpha z(\gamma)d\alpha.
\end{aligned}
$$

In P_1 we can see a part of the R-T condition. The rest of the σ will appear in P_2. We consider $B_i = BR_i(\varpi_1,z)_z + BR_i(\varpi_2,h)_z$, then we split P_2 like the add of the following terms:

$$
Q_1 = \Re\int_{\mathbb{T}} \frac{\Lambda(\overline{\partial^4_\alpha z_1}\partial_\alpha z_2)(\gamma)}{A(t)}B_1\partial^4_\alpha z_1(\gamma)d\alpha,
$$

$$
\begin{aligned}
Q_2 &= \Re \int_{\mathbb{T}} \frac{\Lambda(\overline{\partial_\alpha^4 z_1}\partial_\alpha z_2)(\gamma)}{A(t)} B_2 \partial_\alpha^4 z_2(\gamma) d\alpha,\\
Q_3 &= \Re \int_{\mathbb{T}} \frac{\Lambda(\overline{\partial_\alpha^4 z_2}\partial_\alpha z_1)(\gamma)}{A(t)} B_1 \partial_\alpha^4 z_1(\gamma) d\alpha,\\
Q_4 &= \Re \int_{\mathbb{T}} \frac{\Lambda(\overline{\partial_\alpha^4 z_2}\partial_\alpha z_1)(\gamma)}{A(t)} B_2 \partial_\alpha^4 z_2(\gamma) d\alpha.
\end{aligned}
$$

Using the conmmutator estimations and (2.20),

$$
\begin{aligned}
Q_1 &\leq \exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)} + \|d(z,h)\|^2_{L^\infty(S)} + \|z\|^2_{H^4(S)})\\
&+ \Re \int_{\mathbb{T}} \frac{\partial_\alpha z_2(\gamma) B_1}{A(t)} \partial_\alpha^4 z_1(\gamma) \Lambda(\overline{\partial_\alpha^4 z_1})(\gamma) d\alpha,\\
Q_2 &\leq \exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)} + \|d(z,h)\|^2_{L^\infty(S)} + \|z\|^2_{H^4(S)})\\
&+ \Re \int_{\mathbb{T}} \frac{\partial_\alpha z_2(\gamma) B_2}{A(t)} \partial_\alpha^4 z_2(\gamma) \Lambda(\overline{\partial_\alpha^4 z_1})(\gamma) d\alpha,\\
Q_3 &\leq \exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)} + \|d(z,h)\|^2_{L^\infty(S)} + \|z\|^2_{H^4(S)})\\
&- \Re \int_{\mathbb{T}} \frac{\partial_\alpha z_1(\gamma) B_1}{A(t)} \partial_\alpha^4 z_1(\gamma) \Lambda(\overline{\partial_\alpha^4 z_2})(\gamma) d\alpha,\\
Q_4 &\leq \exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)} + \|d(z,h)\|^2_{L^\infty(S)} + \|z\|^2_{H^4(S)})\\
&- \Re \int_{\mathbb{T}} \frac{\partial_\alpha z_1(\gamma) B_2}{A(t)} \partial_\alpha^4 z_2(\gamma) \Lambda(\overline{\partial_\alpha^4 z_2})(\gamma) d\alpha.
\end{aligned}
$$

Since $\Lambda = H\partial_\alpha$ and $\partial_\alpha z_2(\gamma)\partial_\alpha^4 z(\gamma) = -\partial_\alpha z_1 \partial_\alpha^4 z_1 - 3\partial_\alpha^2 z(\gamma)\cdot\partial_\alpha^3 z(\gamma)$, because of the fact that $A(t) = |\partial_\alpha z(\gamma)|^2$, we get:

$$
\begin{aligned}
Q_2 &\leq \exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)} + \|d(z,h)\|^2_{L^\infty(S)} + \|z\|^2_{H^4(S)})\\
&- \Re \int_{\mathbb{T}} \frac{\partial_\alpha z_1(\gamma) B_2}{A(t)} \partial_\alpha^4 z_1(\gamma) \Lambda(\overline{\partial_\alpha^4 z_1})(\gamma) d\alpha,\\
Q_3 &\leq \exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)} + \|d(z,h)\|^2_{L^\infty(S)} + \|z\|^2_{H^4(S)})\\
&+ \Re \int_{\mathbb{T}} \frac{\partial_\alpha z_2(\gamma) B_1}{A(t)} \partial_\alpha^4 z_2(\gamma) \Lambda(\overline{\partial_\alpha^4 z_2})(\gamma) d\alpha.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\sum_{j=1}^{4} Q_j &\leq \exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)} + \|d(z,h)\|^2_{L^\infty(S)} + \|z\|^2_{H^4(S)})\\
&- \Re \int_{\mathbb{T}} \frac{(BR(\varpi_1,z)_z + BR(\varpi_2,h)_z)\cdot\partial_\alpha^\perp z(\gamma)}{A(t)} \partial_\alpha^4 z(\gamma)\cdot\Lambda(\overline{\partial_\alpha^4 z})(\gamma) d\alpha.
\end{aligned}
$$

Therefore,

$$
P_1 + P_2 \leq \exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)} + \|d(z,h)\|^2_{L^\infty(S)} + \|z\|^2_{H^4(S)})
$$

$$-\Re\int_{\mathbb{T}}\frac{\sigma(\gamma)}{A(t)}\partial_\alpha^4 z(\gamma)\cdot\Lambda(\overline{\partial_\alpha^4 z})(\gamma)d\alpha.$$

Let us to proceed with the last term:

$$\begin{aligned}
&-\Re\int_{\mathbb{T}}\frac{\sigma(\gamma)}{A(t)}\partial_\alpha^4 z(\gamma)\cdot\Lambda(\overline{\partial_\alpha^4 z})(\gamma)d\alpha\\
&=\int_{\mathbb{T}}\Im(\frac{\sigma}{A(t)})(-\Re(\partial_\alpha^4 z)\cdot\Im(\Lambda(\partial_\alpha^4 z))+\Im(\partial_\alpha^4 z)\cdot\Re(\Lambda(\partial_\alpha^4 z)))d\alpha\\
&-\int_{\mathbb{T}}\Re(\frac{\sigma}{A(t)})(\Re(\partial_\alpha^4 z)\cdot\Re(\Lambda(\partial_\alpha^4 z))+\Im(\partial_\alpha^4 z)\cdot\Im(\Lambda(\partial_\alpha^4 z)))d\alpha\equiv Y_1+Y_2,
\end{aligned}$$

we get

$$\begin{aligned}
Y_1&=\int_{\mathbb{T}}(-\Lambda(\Im(\frac{\sigma}{A(t)})\Re(\partial_\alpha^4 z))+\Im(\frac{\sigma}{A(t)})\Re(\Lambda(\partial_\alpha^4 z)))\cdot\Im(\partial_\alpha^4 z)d\alpha\\
&\leq C\|\frac{\sigma}{A(t)}\|_{\mathcal{C}^{1,\delta}(S)}\|\partial_\alpha^4 z\|_{L^2(S)}^2\leq\exp C(\|\mathcal{F}(z)\|_{L^\infty(S)}^2+\|d(z,h)\|_{L^\infty(S)}^2+\|z\|_{H^4(S)}^2)
\end{aligned}$$

and

$$\begin{aligned}
Y_2=&-\int_{\mathbb{T}}(\Re(\frac{\sigma}{A(t)})-m(t))(\Re(\partial_\alpha^4 z)\cdot\Re(\Lambda(\partial_\alpha^4 z))+\Im(\partial_\alpha^4 z)\cdot\Im(\Lambda(\partial_\alpha^4 z)))d\alpha\\
&-\int_{\mathbb{T}}m(t)(\Re(\partial_\alpha^4 z)\cdot\Re(\Lambda(\partial_\alpha^4 z))+\Im(\partial_\alpha^4 z)\cdot\Im(\Lambda(\partial_\alpha^4 z)))d\alpha\\
&\equiv Y_3+Y_4
\end{aligned}$$

where

$$m(t)=\min_{\gamma}\sigma(\gamma,t).$$

Since $\Re(\frac{\sigma}{A(t)})-m(t)>0$ using $2g\Lambda(g)-\Lambda(g^2)\geq 0$ (see [10]) we get

$$\begin{aligned}
Y_3&\leq\frac{1}{2}\|\Lambda(\Re(\frac{\sigma}{A(t)}))\|_{L^\infty(S)}\|\partial_\alpha^4 z\|_{L^2(S)}^2\leq C\|\frac{\sigma}{A(t)}\|_{\mathcal{C}^{1,\delta}(S)}\|\partial_\alpha^4 z\|_{L^2(S)}^2\\
&\leq\exp C(\|\mathcal{F}(z)\|_{L^\infty(S)}^2+\|d(z,h)\|_{L^\infty(S)}^2+\|z\|_{H^4(S)}^2),\\
Y_4&=-m(t)\|\Lambda^{\frac{1}{2}}\partial_\alpha^4 z\|_{L^2(S)}^2.
\end{aligned}$$

Combining all previous estimates

$$I_7\leq\exp C(\|\mathcal{F}(z)\|_{L^\infty(S)}^2+\|z\|_{H^4(S)}^2)-m(t)\|\Lambda^{\frac{1}{2}}\partial_\alpha^4 z\|_{L^2(S)}^2,$$

giving us

$$\begin{aligned}
J_1&\leq\exp C(\|\mathcal{F}(z)\|_{L^\infty(S)}^2+\|d(z,h)\|_{L^\infty(S)}^2+\|z\|_{H^4(S)}^2)\\
&+(C\|\frac{\varpi_1}{A(t)}\|_{H^2(S)}-m(t))\|\Lambda^{\frac{1}{2}}\partial_\alpha^4 z\|_{L^2(S)}^2
\end{aligned}$$

Next, we have to estimate J_2. We split $J_2=I_3^2+I_4^2+I_5^2+I_6^2+I_7^2$ where,

$$I_3^2=\frac{1}{2\pi}\Re\int_{\mathbb{T}}\int_{\mathbb{R}}\overline{\partial_\alpha^4 z(\gamma)}\cdot\partial_\alpha^4(\frac{(z(\gamma)-h(\gamma-\beta))^\perp}{|z(\gamma)-h(\gamma-\beta)|^2})\varpi_2(\gamma-\beta)d\alpha d\beta,$$

$$
\begin{aligned}
I_4^2 &= \frac{2}{\pi}\Re\int_{\mathbb{T}}\int_{\mathbb{R}} \overline{\partial_\alpha^4 z(\gamma)}\cdot\partial_\alpha^3\Big(\frac{(z(\gamma)-h(\gamma-\beta))^\perp}{|z(\gamma)-h(\gamma-\beta)|^2}\Big)\partial_\alpha\varpi_2(\gamma-\beta)d\alpha d\beta,\\
I_5^2 &= \frac{3}{\pi}\Re\int_{\mathbb{T}}\int_{\mathbb{R}} \overline{\partial_\alpha^4 z(\gamma)}\cdot\partial_\alpha^2\Big(\frac{(z(\gamma)-h(\gamma-\beta))^\perp}{|z(\gamma)-h(\gamma-\beta)|^2}\Big)\partial_\alpha^2\varpi_2(\gamma-\beta)d\alpha d\beta,\\
I_6^2 &= \frac{2}{\pi}\Re\int_{\mathbb{T}}\int_{\mathbb{R}} \overline{\partial_\alpha^4 z(\gamma)}\cdot\partial_\alpha\Big(\frac{(z(\gamma)-h(\gamma-\beta))^\perp}{|z(\gamma)-h(\gamma-\beta)|^2}\Big)\partial_\alpha^3\varpi_2(\gamma-\beta)d\alpha d\beta,\\
I_7^2 &= \frac{1}{2\pi}\Re\int_{\mathbb{T}}\int_{\mathbb{R}} \overline{\partial_\alpha^4 z(\gamma)}\cdot\frac{(z(\gamma)-h(\gamma-\beta))^\perp}{|z(\gamma)-h(\gamma-\beta)|^2}\partial_\alpha^4\varpi_2(\gamma-\beta)d\alpha d\beta.
\end{aligned}
$$

We can see easily that $I_4^2+I_5^2+I_6^2\le \exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)}+\|d(z,h)\|^2_{L^\infty(S)}+\|z\|^2_{H^4(S)})$ with the same computations as in Chapter 1. The most singular terms in I_3^2:

$$
\begin{aligned}
I_3^{21} &= \frac{1}{2\pi}\Re\int_{\mathbb{T}}\int_{\mathbb{R}} \overline{\partial_\alpha^4 z(\gamma)}\cdot\frac{(\partial_\alpha^4 z(\gamma)-\partial_\alpha^4 h(\gamma-\beta))^\perp}{|z(\gamma)-h(\gamma-\beta)|^2}\varpi_2(\gamma-\beta)d\alpha d\beta\\
&\le C\|d(z,h)\|_{L^\infty(S)}\|\varpi_2\|_{L^\infty(S)}\|\partial_\alpha^4 z\|^2_{L^2(S)}\\
&+C\|d(z,h)\|_{L^\infty(S)}\|\varpi_2\|_{L^\infty(S)}\|\partial_\alpha^4 h\|_{L^2}\|\partial_\alpha^4 z\|_{L^2(S)},\\
I_3^{22} &= -\frac{1}{\pi}\Re\int_{\mathbb{T}}\int_{\mathbb{R}} \overline{\partial_\alpha^4 z(\gamma)}\cdot\frac{(\Delta zh)^\perp}{|\Delta zh|^4}(\Delta zh\cdot\Delta\partial_\alpha^4 zh)\varpi_2(\gamma-\beta)d\alpha d\beta\\
&\le C\|d(z,h)\|_{L^\infty(S)}\|\varpi_2\|_{L^\infty(S)}\|\partial_\alpha^4 z\|^2_{L^2(S)}\\
&+C\|d(z,h)\|_{L^\infty(S)}\|\varpi_2\|_{L^\infty(S)}\|\partial_\alpha^4 h\|_{L^2}\|\partial_\alpha^4 z\|_{L^2(S)}.
\end{aligned}
$$

Finally for I_7^2,

$$
\begin{aligned}
I_7^2 &= -\frac{1}{2\pi}\Re\int_{\mathbb{T}}\int_{\mathbb{R}} \overline{\partial_\alpha^4 z(\gamma)}\cdot\frac{(z(\gamma)-h(\gamma-\beta))^\perp}{|z(\gamma)-h(\gamma-\beta)|^2}\partial_\beta\partial_\alpha^3\varpi_2(\gamma-\beta)d\alpha d\beta\\
&= \frac{1}{2\pi}\Re\int_{\mathbb{T}}\int_{\mathbb{R}} \overline{\partial_\alpha^4 z(\gamma)}\cdot\frac{\partial_\alpha^\perp h(\gamma-\beta)}{|z(\gamma)-h(\gamma-\beta)|^2}\partial_\alpha^3\varpi_2(\gamma-\beta)d\alpha d\beta\\
&-\frac{1}{2\pi}\Re\int_{\mathbb{T}}\int_{\mathbb{R}} \overline{\partial_\alpha^4 z(\gamma)}\cdot\frac{(\Delta zh)^\perp(\Delta zh\cdot\partial_\alpha h(\gamma-\beta))}{|z(\gamma)-h(\gamma-\beta)|^4}\partial_\alpha^3\varpi_2(\gamma-\beta)d\alpha d\beta\\
&\le C\|d(z,h)\|_{L^\infty(S)}\|\partial_\alpha^3\varpi_2\|_{L^2(S)}\|\partial_\alpha h\|_{L^\infty}\|\partial_\alpha^4 z\|_{L^2(S)}.
\end{aligned}
$$

Therefore,

$$
J_2\le \exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)}+\|d(z,h)\|^2_{L^\infty(S)}+\|z\|^2_{H^4(S)}).
$$

The last term, J_3 is estimated as follows. We split $J_3=I_3^3+I_4^3+I_5^3+I_6^3+I_7^3$, where

$$
\begin{aligned}
I_3^3 &= \Re\int_{\mathbb{T}} \overline{\partial_\alpha^4 z(\gamma)}\cdot\partial_\alpha^4 c(\gamma)\partial_\alpha z(\gamma)d\alpha,\\
I_4^3 &= 4\Re\int_{\mathbb{T}} \overline{\partial_\alpha^4 z(\gamma)}\cdot\partial_\alpha^3 c(\gamma)\partial_\alpha^2 z(\gamma)d\alpha,\\
I_5^3 &= 6\Re\int_{\mathbb{T}} \overline{\partial_\alpha^4 z(\gamma)}\cdot\partial_\alpha^2 c(\gamma)\partial_\alpha^3 z(\gamma)d\alpha,\\
I_6^3 &= 4\Re\int_{\mathbb{T}} \overline{\partial_\alpha^4 z(\gamma)}\cdot\partial_\alpha c(\gamma)\partial_\alpha^4 z(\gamma)d\alpha,\\
I_7^3 &= \Re\int_{\mathbb{T}} \overline{\partial_\alpha^4 z(\gamma)}\cdot c(\gamma)\partial_\alpha^5 z(\gamma)d\alpha.
\end{aligned}
$$

Recall that

$$
\begin{aligned}
c(\alpha,t) &= \frac{\alpha+\pi}{2\pi A(t)}\int_{\mathbb{T}}\partial_\alpha z(\beta,t)\cdot\partial_\alpha(BR(\varpi_1,z)_z+BR(\varpi_2,h)_z)d\beta\\
&-\int_{-\pi}^{\alpha}\frac{\partial_\alpha z(\beta,t)}{A(t)}\cdot\partial_\alpha(BR(\varpi_1,z)_z+BR(\varpi_2,h)_z)d\beta
\end{aligned}
$$

then,

$$
\begin{aligned}
\partial_\alpha^2 c(\alpha) &= -\frac{\partial_\alpha^2 z(\alpha)}{A(t)}\cdot\partial_\alpha(BR(\varpi_1,z)_z+BR(\varpi_2,h)_z)\\
&-\frac{\partial_\alpha z(\alpha)}{A(t)}\cdot\partial_\alpha^2(BR(\varpi_1,z)_z+BR(\varpi_2,h)_z)
\end{aligned}
$$

and

$$
\begin{aligned}
\partial_\alpha^3 c(\alpha,t) &= -\frac{\partial_\alpha^3 z(\alpha)}{A(t)}\cdot\partial_\alpha(BR(\varpi_1,z)_z+BR(\varpi_2,h)_z)\\
&-\frac{2\partial_\alpha^2 z(\alpha,t)}{A(t)}\cdot\partial_\alpha^2(BR(\varpi_1,z)_z+BR(\varpi_2,h)_z)\\
&-\frac{\partial_\alpha z(\alpha)}{A(t)}\cdot\partial_\alpha^3(BR(\varpi_1,z)_z+BR(\varpi_2,h)_z).
\end{aligned}
$$

Using (2.20) directly we have I_4^3 and I_5^3 controlled. For I_6^3,

$$
\begin{aligned}
I_6^3 &\le C\|\partial_\alpha c\|_{L^\infty(S)}\|\partial_\alpha^4 z\|_{L^2}^2\\
&\le C\|\mathcal{F}(z)\|_{L^\infty(S)}^{\frac{1}{2}}\|\partial_\alpha(BR(\varpi_1,z)_z+BR(\varpi_2,h)_z)\|_{L^\infty(S)}\|\partial_\alpha^4 z\|_{L^2(S)}^2.
\end{aligned}
$$

For I_7^3 we need to split,

$$
\begin{aligned}
I_7^3 &= \int_{\mathbb{T}}\Re(c)(\Re(\partial_\alpha^4 z)\Re(\partial_\alpha^5 z)+\Im(\partial_\alpha^4 z)\Im(\partial_\alpha^5 z))d\alpha\\
&+\int_{\mathbb{T}}\Im(c)(-\Re(\partial_\alpha^4 z)\Im(\partial_\alpha^5 z)+\Im(\partial_\alpha^4 z)\Re(\partial_\alpha^5 z))d\alpha \equiv I_7^{31}+I_7^{32}.
\end{aligned}
$$

A wise integrating by parts,

$$
I_7^{31} = -\frac{1}{2}\int_{\mathbb{T}}\Re(\partial_\alpha c)|\partial_\alpha^4 z|^2 d\alpha \le \|\partial_\alpha c\|_{L^\infty(S)}\|\partial_\alpha^4 z\|_{L^2(S)}^2
$$

and

$$
\begin{aligned}
I_7^{32} &= \int_{\mathbb{T}}\Im(\partial_\alpha c)\Re(\partial_\alpha^4 z)\Im(\partial_\alpha^4 z)d\alpha + 2\int_{\mathbb{T}}\Im(c)\Im(\partial_\alpha^4 z)\Re(\partial_\alpha^5 z)d\alpha\\
&\le \|\partial_\alpha c\|_{L^\infty(S)}\|\partial_\alpha^4 z\|_{L^2(S)}^2 - 2\int_{\mathbb{T}}\Im(c)\Im(\partial_\alpha^4 z)\Re(\Lambda(H(\partial_\alpha^4 z)))d\alpha\\
&\le \exp C(\|\mathcal{F}(z)\|_{L^\infty(S)}^2+\|d(z,h)\|_{L^\infty(S)}^2+\|z\|_{H^4(S)}^2)\\
&-2\int_{\mathbb{T}}\Lambda^{\frac{1}{2}}(\Im(c)\Im(\partial_\alpha^4 z))\Re(\Lambda^{\frac{1}{2}}(H(\partial_\alpha^4 z)))d\alpha
\end{aligned}
$$

$$\leq \exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)} + \|d(z,h)\|^2_{L^\infty(S)} + \|z\|^2_{H^4(S)})$$
$$+ C\|\Im(c)\|_{H^2(S)}\|\Lambda^{\frac{1}{2}}\partial^4_\alpha z\|^2_{L^2(S)}.$$

Finally, since

$$\partial^4_\alpha c(\alpha,t) = -\frac{\partial^4_\alpha z(\alpha)}{A(t)}\cdot\partial_\alpha(BR(\varpi_1,z)_z + BR(\varpi_2,h)_z)$$
$$-\frac{3\partial^3_\alpha z(\alpha,t)}{A(t)}\cdot\partial^2_\alpha(BR(\varpi_1,z)_z + BR(\varpi_2,h)_z)$$
$$-\frac{3\partial^2_\alpha z(\alpha)}{A(t)}\cdot\partial^3_\alpha(BR(\varpi_1,z)_z + BR(\varpi_2,h)_z)$$
$$-\frac{\partial_\alpha z(\alpha)}{A(t)}\cdot\partial^4_\alpha(BR(\varpi_1,z)_z + BR(\varpi_2,h)_z) \equiv C_1(\alpha)+C_2(\alpha)+C_3(\alpha)+C_4(\alpha),$$

we separate $I^3_3 = N_6 + N_7 + N_8 + N_9$:

$$N_6 = -\Re\int_{\mathbb{T}}\frac{\overline{\partial^4_\alpha z(\gamma)}}{A(t)}\cdot\partial_\alpha z(\gamma)C_1(\gamma)d\alpha,$$
$$N_7 = -\Re\int_{\mathbb{T}}\frac{\overline{\partial^4_\alpha z(\gamma)}}{A(t)}\cdot\partial_\alpha z(\gamma)C_2(\gamma)d\alpha,$$
$$N_8 = -\Re\int_{\mathbb{T}}\frac{\overline{\partial^4_\alpha z(\gamma)}}{A(t)}\cdot\partial_\alpha z(\gamma)C_3(\gamma)d\alpha,$$
$$N_9 = -\Re\int_{\mathbb{T}}\frac{\overline{\partial^4_\alpha z(\gamma)}}{A(t)}\cdot\partial_\alpha z(\gamma)C_4(\gamma)d\alpha.$$

Using (2.20) we get controlled $N_6 + N_7 + N_9$. We can write

$$N_9 = -\Re\int_{\mathbb{T}}\frac{\overline{\partial^4_\alpha z(\gamma)}}{A(t)}\cdot\partial_\alpha z(\gamma)\partial_\alpha z(\gamma)\cdot\partial^4_\alpha BR(\varpi_1,z)_z(\gamma)d\alpha$$
$$-\Re\int_{\mathbb{T}}\frac{\overline{\partial^4_\alpha z(\gamma)}}{A(t)}\cdot\partial_\alpha z(\gamma)\partial_\alpha z(\gamma)\cdot\partial^4_\alpha BR(\varpi_2,h)_z(\gamma)d\alpha \equiv N^1_9 + N^2_9.$$

The term N^1_9 is exactly the term N_9 in section 1.1.1.3. Therefore,

$$N^1_9 \leq \exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)} + \|d(z,h)\|^2_{L^\infty(S)} + \|z\|^2_{H^4(S)})$$
$$+ C(\|\Im(\partial_\alpha z)\Re(\partial_\alpha z\frac{\varpi_1}{A(t)})\|_{H^2(S)} + \|\Im(\partial_\alpha z)\Im(\partial_\alpha z\frac{\varpi_1}{A(t)})\|_{H^2(S)})\|\Lambda^{\frac{1}{2}}\partial^4_\alpha z\|^2_{L^2(S)}.$$

Then, we will study the term N^2_9:

$$N^{21}_9 = -\Re\int_{\mathbb{T}}\int_{\mathbb{R}}\frac{\overline{\partial^4_\alpha z(\gamma)}}{A(t)}\cdot\partial_\alpha z(\gamma)\partial_\alpha z(\gamma)\cdot\partial^4_\alpha(\frac{(z(\gamma)-h(\gamma-\beta))^\perp}{|z(\gamma)-h(\gamma-\beta)|^2})\varpi_2(\gamma-\beta)d\alpha d\beta,$$
$$N^{22}_9 = -4\Re\int_{\mathbb{T}}\int_{\mathbb{R}}\frac{\overline{\partial^4_\alpha z(\gamma)}}{A(t)}\cdot\partial_\alpha z(\gamma)\partial_\alpha z(\gamma)\cdot\partial^3_\alpha(\frac{(z(\gamma)-h(\gamma-\beta))^\perp}{|z(\gamma)-h(\gamma-\beta)|^2})\partial_\alpha\varpi_2(\gamma-\beta)d\alpha d\beta,$$

$$
\begin{aligned}
N_9^{23} &= -6\Re\int_{\mathbb{T}}\int_{\mathbb{R}} \frac{\overline{\partial_\alpha^4 z(\gamma)}}{A(t)}\cdot\partial_\alpha z(\gamma)\partial_\alpha z(\gamma)\cdot\partial_\alpha^2\Big(\frac{(z(\gamma)-h(\gamma-\beta))^\perp}{|z(\gamma)-h(\gamma-\beta)|^2}\Big)\partial_\alpha^2\varpi_2(\gamma-\beta)d\alpha d\beta,\\
N_9^{24} &= -4\Re\int_{\mathbb{T}}\int_{\mathbb{R}} \frac{\overline{\partial_\alpha^4 z(\gamma)}}{A(t)}\cdot\partial_\alpha z(\gamma)\partial_\alpha z(\gamma)\cdot\partial_\alpha\Big(\frac{(z(\gamma)-h(\gamma-\beta))^\perp}{|z(\gamma)-h(\gamma-\beta)|^2}\Big)\partial_\alpha^3\varpi_2(\gamma-\beta)d\alpha d\beta,\\
N_9^{25} &= -\Re\int_{\mathbb{T}}\int_{\mathbb{R}} \frac{\overline{\partial_\alpha^4 z(\gamma)}}{A(t)}\cdot\partial_\alpha z(\gamma)\partial_\alpha z(\gamma)\cdot\frac{(z(\gamma)-h(\gamma-\beta))^\perp}{|z(\gamma)-h(\gamma-\beta)|^2}\partial_\alpha^4\varpi_2(\gamma-\beta)d\alpha d\beta.
\end{aligned}
$$

With the same procedure that in the well-posedness estudy, $N_9^{22}+N_9^{23}+N_{24}\leq \exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)}+\|d(z,h)\|^2_{L^\infty(S)}+\|z\|^2_{H^4(S)})$.

The most singular terms in N_9^{21} are:

$$
\begin{aligned}
N_9^{211} &= -\Re\int_{\mathbb{T}}\int_{\mathbb{R}} \frac{\overline{\partial_\alpha^4 z(\gamma)}}{A(t)}\cdot\partial_\alpha z(\gamma)\partial_\alpha z(\gamma)\cdot\frac{(\partial_\alpha^4 z(\gamma)-\partial_\alpha^4 h(\gamma-\beta))^\perp}{|z(\gamma)-h(\gamma-\beta)|^2}\varpi_2(\gamma-\beta)d\alpha d\beta,\\
N_9^{212} &= -\Re\int_{\mathbb{T}}\int_{\mathbb{R}} \frac{\overline{\partial_\alpha^4 z(\gamma)}}{A(t)}\cdot\partial_\alpha z(\gamma)\partial_\alpha z(\gamma)\cdot\frac{(\Delta zh)^\perp\Delta zh\cdot\Delta\partial_\alpha^4 zh}{|z(\gamma)-h(\gamma-\beta)|^4}\varpi_2(\gamma-\beta)d\alpha d\beta.
\end{aligned}
$$

Therefore,

$$
N_9^{211}+N_9^{212}\leq C\|\partial_\alpha^4 z\|_{L^2(S)}\|d(z,h)\|_{L^\infty(S)}(\|\partial_\alpha^4 z\|_{L^2(S)}+\|\partial_\alpha^4 h\|_{L^2(S)})\|\varpi_2\|_{L^\infty(S)}.
$$

Using integration by parts,

$$
\begin{aligned}
N_9^{25} &= -\Re\int_{\mathbb{T}}\int_{\mathbb{R}} \frac{\overline{\partial_\alpha^4 z(\gamma)}}{A(t)}\cdot\partial_\alpha z(\gamma)\partial_\alpha z(\gamma)\cdot\partial_\beta\Big(\frac{(z(\gamma)-h(\gamma-\beta))^\perp}{|z(\gamma)-h(\gamma-\beta)|^2}\Big)\partial_\alpha^3\varpi_2(\gamma-\beta)d\alpha d\beta\\
&\leq C\|\partial_\alpha^4 z\|_{L^2(S)}\|d(z,h)\|_{L^\infty(S)}(\|z\|_{\mathcal{C}^1(S)}+\|h\|_{\mathcal{C}^1(S)})\|\partial_\alpha^3\varpi_2\|_{L^\infty(S)}.
\end{aligned}
$$

In conclusion,

$$
N_9^2\leq \exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)}+\|d(z,h)\|^2_{L^\infty(S)}+\|z\|^2_{H^4(S)}).
$$

Taking in to account all previous estimate,

$$
\begin{aligned}
&J_1+J_2+J_3\leq \exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)}+\|d(z,h)\|^2_{L^\infty(S)}+\|z\|^2_{H^4(S)})\\
&+C(\|\Im(c)\|_{H^2(S)}+\|\Im(\partial_\alpha z)\Re(\partial_\alpha z\frac{\varpi_1}{A(t)})\|_{H^2(S)}+\|\Im(\partial_\alpha z)\Im(\partial_\alpha z\frac{\varpi_1}{A(t)})\|_{H^2(S)}\\
&+\|\Im(\frac{\varpi_1}{A(t)})\|_{H^2(S)}-m(t))\|\Lambda^{\frac12}\partial_\alpha^4 z\|^2_{L^2(S)}.
\end{aligned}
$$

If we denote $|||f|||=\|\Im(c)\|_{H^2(S)}+\|\Im(\partial_\alpha z)\Re(\partial_\alpha z\frac{\varpi_1}{A(t)})\|_{H^2(S)}+\|\Im(\partial_\alpha z)\Im(\partial_\alpha z\frac{\varpi_1}{A(t)})\|_{H^2(S)}+\|\Im(\frac{\varpi_1}{A(t)})\|_{H^2(S)}$, we get

$$
\begin{aligned}
&\frac{d}{dt}\|z\|^2_{H^4(S)}\leq \exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)}+\|d(z,h)\|^2_{L^\infty(S)}+\|z\|^2_{H^4(S)})\\
&+C(|||f|||+2\lambda-m(t))\|\Lambda^{\frac12}\partial_\alpha^4 z\|_{L^2(S)}.
\end{aligned}
$$

Note that $|||f|||(0) = 0$. If $2\lambda - m(0) < 0$, for small time

$$|||f|||(t) + 2\lambda - m(t) < 0.$$

While this is true,

$$\frac{d}{dt}\|z\|^2_{H^4(S)} \leq \exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)} + \|d(z,h)\|^2_{L^\infty(S)} + \|z\|^2_{H^4(S)})$$

Proceeding like in sections 2.6, 2.7 and 2.8 in Chapter 2:

$$\begin{aligned}
&\frac{d}{dt}\|\mathcal{F}(z)\|_{L^\infty(S)} \leq \exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)} + \|d(z,h)\|^2_{L^\infty(S)} + \|z\|^2_{H^4(S)}),\\
&\frac{d}{dt}\|d(z,h)\|_{L^\infty(S)} \leq \exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)} + \|d(z,h)\|^2_{L^\infty(S)} + \|z\|^2_{H^4(S)}),\\
&m(t) \geq m(0) - \int_0^t \exp C|||z,h|||^2(s)ds.
\end{aligned}$$

We denote our energy by,

$$E(z) = \|\mathcal{F}(z)\|^2_{L^\infty(S)} + \|d(z,h)\|^2_{L^\infty} + \|z\|^2_{H^4(S)} + \frac{1}{m(t) - 2\lambda - |||f|||(t)}.$$

It is easy to prove that $|||f|||(t) \leq \exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)} + \|d(z,h)\|^2_{L^\infty(S)} + \|z\|^2_{H^4(S)})$ and

$$\frac{d}{dt}\left(\frac{1}{m(t) - 2\lambda - |||f|||(t)}\right) \leq \frac{\exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)} + \|d(z,h)\|^2_{L^\infty(S)} + \|z\|^2_{H^4(S)})}{m(t) - 2\lambda - |||f|||(t)}.$$

Therefore,

$$\frac{d}{dt}E(z) \leq \exp CE(z)(t).$$

Now we follow the classical regularization of the problem using the heat kernel. Picard's theorem yields the existence and analitycity of the problem. ■

3.1.2 Control of the strip of analyticity

The following theorem give us the control of the strip of analyticity:

Theorem 3.1.2. *Let $z(\alpha, 0) = z^0(\alpha)$ be an analytic curve in the strip*

$$S = \{\alpha + i\varsigma \in \mathbb{C} : |\varsigma| < f(0)\},$$

with $f(0) > 0$, $h(\alpha) \in H^4(\mathbb{T})$, $\d(z^0, h) \in L^\infty(\mathbb{T})$, $\mathcal{F}(h) \in L^\infty(\mathbb{T})$ and satisfying:

* *The arc-chord condition, $\mathcal{F}(z^0)(\alpha + i\varsigma, \beta) \in L^\infty(S \times \mathbb{R})$,*
* *The Rayleigh-Taylor condition, $\sigma(\alpha, 0) > 0$,*
* *The curve $z^0(\alpha)$ is real for real α,*
* *The functions $z_1^0(\alpha) - \alpha$ and $z_2{}^0(\alpha)$ are periodic with period 2π,*
* *The functions $z_1{}^0(\alpha) - \alpha$ and $z_2{}^0(\alpha)$ belong to $H^4(S)$.*

Then there exist a time T and a solution of the Muskat problem $z(\alpha,t)$ defined for $0<t\leq T$ that continues analytically into some complex strip for each fixed $t\in[0,T]$. Here T is either a small constant depending only on $\exp C(\|\mathcal{F}(z^0)\|^2_{L^\infty(S)}+\|d(z^0,h)\|_{L^\infty(S)}+\|z^0\|^2_{H^4(S)})$.

Proof. Here we only have to follow the proof of the theorem 1.4.1 in 1.4. The difference is that we have to consider

$$J[z](\alpha,t)=BR(\varpi_1,z)_z(\alpha,t)+BR(\varpi_2,h)_z(\alpha,t)+c(\alpha,t)\partial_\alpha z(\alpha,t).$$

Then,

$$\begin{aligned}
&\frac{d}{dt}\sum_{\pm}\int_{\mathbb{T}}|\partial^4_\alpha z_j(\alpha\pm if(t))|^2d\alpha\leq\frac{f'(t)}{10}\sum_{\pm}\int_{\mathbb{T}}\Lambda(\partial^4_\alpha z_j)(\alpha\pm if(t))\overline{\partial^4_\alpha z_j(\alpha\pm if(t))}\\
&-10h'(t)\int_{\mathbb{T}}\Lambda(\partial^4_\alpha z_j)(\alpha)\overline{\partial^4_\alpha z_j(\alpha)}d\alpha+2\sum_{\pm}\Re\int_{\mathbb{T}}\partial^4_\alpha B(\alpha\pm if(t))\overline{\partial^4_\alpha z_j(\alpha\pm if(t))}\\
&+2\sum_{\pm}\Re\int_{\mathbb{T}}\partial^4_\alpha(c(\alpha\pm if(t))\partial_\alpha z_j(\alpha\pm if(t)))\overline{\partial^4_\alpha z_j(\alpha\pm ih(t))}\\
&\equiv M_1+M_2+M_3+M_4,
\end{aligned}$$

where $B=BR(\varpi_1,z)_z+BR(\varpi_2,h)_z$ Since we already have estimate $z(\gamma)$, we repeat these estimation and we will get,

$$M_4\leq\exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)}+\|d(z,h)\|^2_{L^\infty(S)}+\|z\|^2_{H^4(S)})\|\Lambda^{\frac{1}{2}}\partial^4_\alpha z\|^2_{L^2(S)}$$

and

$$\begin{aligned}
M_3\leq&\exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)}+\|d(z,h)\|^2_{L^\infty(S)}+\|z\|^2_{H^4(S)})\\
&+C\|\Im(\frac{\varpi_1}{A(t)})\|_{H^2(S)}\|\Lambda^{\frac{1}{2}}\partial^4_\alpha z\|^2_{L^2(S)}-2\Re\int_{\mathbb{T}}\frac{\sigma(\gamma)}{A(t)}\partial^4_\alpha z(\gamma)\cdot\Lambda^{\frac{1}{2}}(\overline{\partial^4_\alpha z})(\gamma)d\alpha.
\end{aligned}$$

In order to avoid problems we write,

$$\sigma(\gamma)=\sigma(\alpha)+f(t)g_\pm(\alpha)$$

where $g_\pm=\frac{1}{f(t)}(\sigma(\gamma)-\sigma(\alpha))$.

Since

$$\sigma(\alpha)=\frac{\mu^2}{\kappa^1}B(\alpha)\cdot\partial^\perp_\alpha z(\alpha)+g\rho^2\partial_\alpha z_1(\alpha),$$

we can write,

$$g_\pm=\pm\frac{i\mu^2}{\kappa^1}\int_0^1\partial_\alpha(B\cdot\partial^\perp_\alpha z)(\gamma t+(t-1)\alpha)dt\pm ig\rho^2\int_0^1\partial^2_\alpha z_1(\gamma t+(t-1)\alpha)dt$$

then

$$\|g_\pm\|_{H^2(S)}\leq\exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)}+\|d(z,h)\|^2_{L^\infty(S)}+\|z\|^2_{H^4(S)}).$$

Therefore,

$$M_3\leq\exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)}+\|d(z,h)\|^2_{L^\infty(S)}+\|z\|^2_{H^4(S)})$$

$$+ f(t)\exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)} + \|d(z,h)\|^2_{L^\infty(S)} + \|z\|^2_{H^4(S)})\|\Lambda^{\frac{1}{2}}\partial^4_\alpha z\|^2_{L^2(S)}.$$

Since,

$$M_1 \leq \frac{f'(t)}{10}\|\Lambda^{\frac{1}{2}}\partial^4_\alpha z\|^2_{L^2(S)},$$

Then,

$$\begin{aligned}&\frac{d}{dt}\sum_{\pm}\int_{\mathbb{T}}|\partial^4_\alpha z_j(\alpha \pm if(t))|^2 d\alpha \leq \exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)} + \|d(z,h)\|^2_{L^\infty(S)} + \|z\|^2_{H^4(S)})\\&- 10f'(t)\int_{\mathbb{T}}\Lambda(\partial^4_\alpha z_j)(\alpha)\overline{\partial^4_\alpha z_j(\alpha)}d\alpha + (\exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)} + \|d(z,h)\|^2_{L^\infty(S)} + \|z\|^2_{H^4(S)})f(t)\\&+ \frac{f'(t)}{10} + \exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)} + \|d(z,h)\|^2_{L^\infty(S)} + \|z\|^2_{H^4(S)}))\|\Lambda^{\frac{1}{2}}\partial^4_\alpha z\|^2_{L^2(S)}.\end{aligned}$$

Choosing,

$$f(t) = \exp(-10\int_0^t G(r)dr)[\int_0^t -10G(r)\exp(10\int_0^r G(s)ds)dr + h(0)]$$

where $G(t) = \exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)} + \|d(z,h)\|^2_{L^\infty(S)} + \|z\|^2_{H^4(S)})(t)$, we eliminate the most dangerous term and we can finish the proof. ■

3.1.3 Non-splat idea

The idea of the contradiction argument is the following:

Suppose that there exists a time T where we have a splat singularity, i.e., the smooth interface collapses along an arc at time T.

Initially the curve is real, by theorem 3.1.1, it instantly becomes analytic. From theorem 3.1.2, our strip of analyticity is nonzero as long as the regularity of the curve and the arc-chord condition not fail.

In Ω domain, at splat time T, the arc-chord condition blow-up so we can not guarantee analyticity at that time. In order to get around this issue it is necessary to apply a transformation P which is a conformal map (see [4]):

$$P(w) = (\tan(\frac{w}{2}))^{\frac{1}{2}}.$$

This conformal map transforms our domain Ω in $\tilde{\Omega}$ as we can see in Figure 3.1. The branch of the root will be taken in such a way that it separates the self-intersecting points of the interface.

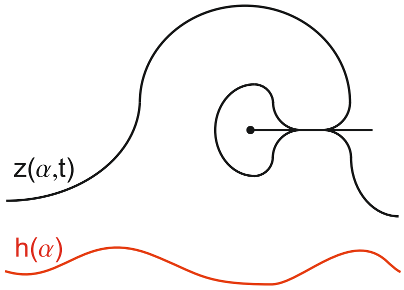
The equations of the inhomogeneous one-phase Muskat problem when we apply the conformal map P are:

$$\tilde{z}_t(\alpha,t) = Q^2(\alpha,t)(BR(\tilde{\varpi}_1,\tilde{z})_{\tilde{z}} + BR(\tilde{\varpi}_2,\tilde{h})_{\tilde{z}})(\alpha,t) + \tilde{c}(\alpha)\partial_\alpha\tilde{z}(\alpha,t) \tag{3.3}$$

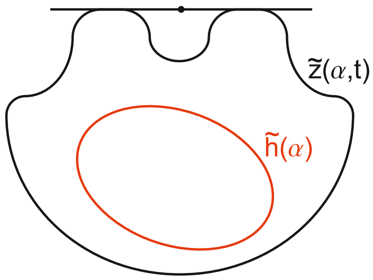
where

$$Q^2(\alpha,t) = |\frac{dP}{dw}(z(\alpha,t))|^2 = |\frac{dP}{dw}(P^{-1}(\tilde{z}(\alpha,t)))|^2, \tag{3.4}$$

$$\tilde{\varpi}_1(\alpha,t) = -2(BR(\tilde{\varpi}_1,\tilde{z})_{\tilde{z}} + BR(\tilde{\varpi}_2,\tilde{h})_{\tilde{z}})(\alpha,t)\cdot\partial_\alpha\tilde{z}(\alpha,t) - 2\frac{g\rho^2\kappa^1}{\mu^2}\partial_\alpha(P_2^{-1}(\tilde{z}(\alpha,t))) \tag{3.5}$$



(a) $\Omega(T)$ domain



(b) $\tilde{\Omega}(T)$ domain

Figure 3.1: Finite time singularities

$$\tilde{\varpi}_2(\alpha,t) = -2\frac{\kappa^2-\kappa^1}{\kappa^1+\kappa^2}(BR(\tilde{\varpi}_1,\tilde{z})_{\tilde{z}} + BR(\tilde{\varpi}_2,\tilde{h})_{\tilde{z}})(\alpha,t)\cdot\partial_\alpha\tilde{h}(\alpha,t) \tag{3.6}$$

and

$$\begin{aligned}\tilde{c}(\alpha,t) &= \frac{\alpha+\pi}{2\pi}\int_{\mathbb{T}}\frac{\partial_\beta\tilde{z}(\beta,t)}{|\partial_\beta\tilde{z}(\beta,t)|^2}\cdot\partial_\beta(Q^2(BR(\tilde{\varpi}_1,\tilde{z})_{\tilde{z}} + BR(\varpi_2,h)_z))(\beta,t)d\beta\\ &- \int_{-\pi}^{\alpha}\frac{\partial_\beta\tilde{z}(\beta,t)}{|\partial_\beta\tilde{z}(\beta,t)|^2}\cdot\partial_\beta(Q^2(BR(\varpi_1,z)_z + BR(\varpi_2,h)_z))(\beta,t)d\beta \end{aligned}\tag{3.7}$$

To find Rayleigh-Taylor condition we define $\tilde{p}(\tilde{x},t) = p(x,t)$. Using the Darcy's law:

$$-\nabla\tilde{p}(\tilde{x},t) = \frac{\nu^2}{\kappa^1}\nabla\tilde{\phi}(\tilde{x},t) + g\rho^2\nabla P_2^1(\tilde{x}).$$

If we approximate to the curve $\tilde{z}(\alpha,t)$:

$$\begin{aligned}\tilde{\sigma}(\alpha,t) &= -\nabla\tilde{p}(\tilde{z}(\alpha,t),t)\cdot\partial_\alpha^\perp\tilde{z}(\alpha,t)\\ &= \frac{\mu^2}{\kappa^1}(BR(\tilde{\varpi}_1,\tilde{z})_{\tilde{z}} + BR(\tilde{\varpi}_2,\tilde{h})_{\tilde{z}})(\alpha,t)\cdot\partial_\alpha^\perp\tilde{z}(\alpha,t) + \rho^2 g\nabla P_2^{-1}(\tilde{z}(\alpha,t))\cdot\partial_\alpha^\perp\tilde{z}(\alpha,t).\end{aligned}$$

The Rayleigh-Taylor condition is given by:

$$\tilde{\sigma}(\alpha,t) = \frac{\mu^2}{\kappa^1}(BR(\tilde{\varpi}_1,\tilde{z})_{\tilde{z}} + BR(\tilde{\varpi}_2,\tilde{h})_{\tilde{z}})(\alpha,t)\cdot\partial_\alpha^\perp\tilde{z}(\alpha,t) + \rho^2 g\nabla P_2^{-1}(\tilde{z}(\alpha,t))\cdot\partial_\alpha^\perp\tilde{z}(\alpha,t) > 0.$$

For more details about how to derive these equations see [3].

In this new domain, we can conclude that there exists a solution of the transformed problem $\tilde{z}(\alpha,t)$ defined for $0 < t \leq \tilde{T}$ that continue analytically into the strip $S(t) = \{\alpha \pm i\varsigma t : |\varsigma| < \lambda t\}$ for each t. And this complex strip decays exponentially to a time that depends on the regularity of the curve and the arc-chord condition.

If we assume that there exists a time $T \in [0,\tilde{T}]$ such that there has a splat singularity in $\tilde{\Omega}$, since in this domain the arc-chord condition and the regularity of the curve are bounded, the strip of analyticity is nonzero and therefore we have analyticity guarantee at the time T.

Thus, applying P^{-1}, we have the analytic curve self-intersects along an arc, therefore we get a contradiction and the non-splat is proved.

Therefore we only have to prove the instant analyticity and the control of the strip of analyticity for the new domain $\tilde{\Omega}$ of the equations (3.3)-(3.7).

The technical computations of these proofs in this new domain are the same as the followed in section 1.5. The sketch is the following:

We only have to prove energy estimates in $\tilde{\Omega}$ for solutions $\tilde{z} \in \mathcal{C}([0,T], H^k)$ for $k \geq 4$.

We define

$$q^0 = (0,0), \quad q^1 = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}), \quad q^2 = (-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}),$$
$$q^3 = (-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}), \quad q^4 = (\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$$

which are the singular points of the P^{-1} conformal map. We set $z(\alpha,t)$ and $h(\alpha,t)$ to hold $\tilde{z}(\alpha,t) \neq q^l$ and $\tilde{h}(\alpha,t) \neq q^l$ for $l = 0,1,2,3,4$. In order to get this we fix $\overline{\Omega(0)}$ so that $\frac{dP}{dw}(w) \neq 0$ for any $w \in \overline{\Omega(0)}$ without loss of generality. We define the energy

$$\|\tilde{z}\|_{RT} \equiv \|\tilde{z}\|^2_{H^k(S)} + \|\mathcal{F}(\tilde{z})\|^2_{L^\infty(S)} + \|d(\tilde{z},\tilde{h})\|^2_{L^\infty} + \frac{1}{m(Q^2\tilde{\sigma})(t) - 2\lambda - \|g\|(t)} + \sum_{l=0}^{4} \frac{1}{m(q^l)(t)}$$

where

$$\|g\|(t) = C(\|\Im(\partial_\alpha \tilde{z})\Re(\partial_\alpha \tilde{z} \frac{\tilde{\varpi} Q^2}{A^2(t)})\|_{H^2(S)} + \|\Im(\partial_\alpha \tilde{z})\Im(\partial_\alpha \tilde{z}\frac{\tilde{\varpi} Q^2}{A^2(t)})\|_{H^2(S)}$$
$$+ \|\Im(\frac{\tilde{\varpi} Q^2}{A(t)})\|_{H^2(S)} + \|\Im(\tilde{c})\|_{H^2(S)})$$

and

$$m(Q^2\tilde{\sigma})(t) = \min_\alpha Q^2(\alpha,t)\tilde{\sigma}(\alpha,t), \quad m(q^l)(t) = \min_\alpha |\tilde{z}(\alpha,t) - q^l|.$$

If we repeat all estimates in section 3.1.1, we will get that: Let $\tilde{z}(\alpha,t)$ be a solution of (3.3-3.7). Then, the following estimate holds:

$$\frac{d}{dt}\|\tilde{z}\|_{RT} \leq \exp C(\|\tilde{z}\|_{RT})$$

for C constant.

The only difference which we will observe with respect to section 3.1.1 is that the factor Q^2 appears. But this factor do not introduce a high order term, it is at the same level than z (See [4] for dealing with Q^2 term):

$$\|Q^2\|_{H^k(S)} \leq \exp C(\|\tilde{z}\|_{RT}).$$

For the control of the strip of analyticity we do the same as in Theorem 3.1.2. Taking

$$Q^2(\gamma)\tilde{\sigma}(\gamma) = Q^2(\alpha)\tilde{\sigma}(\alpha) + f(t)\tilde{g_\pm}(\alpha)$$

and

$$h(t) = \exp(-10\int_0^t G(r)dr)[\int_0^t -10G(r)\exp(10\int_0^r G(s)ds)dr + h(0)]$$

where $G(t) = \exp C(\|\mathcal{F}(\tilde{z})\|^2_{L^\infty(S)} + \|d(\tilde{z},\tilde{h})\|^2_{L^\infty(S)} + \|\tilde{z}\|^2_{H^4(S)})(t)$ we get the desired estimation.

3.2 Splash singularity for the one-phase inhomogeneous Muskat problem

3.2.1 The family of curves z^l

We may consider the family of curves $z^l(\alpha,t)$ such that they have a splash singularity in

$$x_s = z^l(\alpha_1) = z^l(\alpha_2)$$

with $\alpha_1 \neq \alpha_2$ where $\partial_\alpha z_1^l(\alpha_1) = \partial_\alpha z_1^l(\alpha_2) = 0$.

Plugging these curves in Darcy's law,

$$\frac{\mu^2}{\kappa^1} u(z^l(\alpha,t)) = -\nabla p(z^l(\alpha,t),t) - g\rho^2(0,1).$$

Now we multiply by $\partial_\alpha^\perp z^l(\alpha,t)$,

$$\frac{\mu^2}{\kappa^1} u(z^l(\alpha,t)) \cdot \partial_\alpha^\perp z^l(\alpha,t) = -\nabla p(z^l(\alpha,t),t) \cdot \partial_\alpha^\perp z^l(\alpha,t) - g\rho^2 \partial_\alpha z_1^l(\alpha,t),$$

since $\partial_\alpha z_1^l(\alpha_i,t) = 0$ for $i = 1,2$ we have

$$u(z^l(\alpha_i,t)) \cdot \partial_\alpha^\perp z^l(\alpha_i,t) = -\frac{\kappa^1}{\mu^2} \nabla p(z^l(\alpha_i,t),t) \cdot \partial_\alpha^\perp z^l(\alpha_i,t) > 0 \tag{3.8}$$

because of the Rayleigh-Taylor condition. This give us a sign for the velocity at x_s, it is clear that (3.8) implies that the velocity separates the splash point backwards in time.

3.2.2 Local-existence in the tilde domain

We define

$$\begin{aligned} &q^0 = (0,0), \quad q^1 = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}), \quad q^2 = (-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}), \\ &q^3 = (-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}), \quad q^4 = (\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}) \end{aligned}$$

which are the singular points of the P^{-1} conformal map. We set $z(\alpha,t)$ and $h(\alpha)$ to hold $\tilde{z}(\alpha,t) \neq q^l$ and $\tilde{h}(\alpha,t) \neq q^l$ for $l = 0,1,2,3,4$. In order to get this we fix $\overline{\Omega(0)}$ so that $\frac{dP}{dw}(w) \neq 0$ for any $w \in \overline{\Omega(0)}$ without loss of generality. We define the energy

$$E_k(\tilde{z})(t) \equiv \|\tilde{z}\|_{H^k}^2 + \|\mathcal{F}(\tilde{z})\|_{L^\infty}^2 + \|d(\tilde{z},\tilde{h})\|_{L^\infty}^2 + \frac{1}{m(Q^2\tilde{\sigma})(t)} + \sum_{l=0}^{4} \frac{1}{m(q^l)(t)}$$

where

$$\begin{aligned} \mathcal{F}(z) &= \frac{|\beta|}{|z(\alpha,t) - z(\alpha-\beta,t)|^2}, \\ d(z,h) &= \frac{1}{|z(\alpha,t) - h(\alpha-\beta)|^2}. \end{aligned}$$

and

$$m(Q^2\tilde{\sigma}) = \min_{\alpha} Q^2(\alpha,t)\tilde{\sigma}(\alpha,t), \quad m(q^l) = \min_{\alpha}(|z(\alpha,t) - q^l|)$$

Theorem 3.2.1. *The following estimation holds:*

$$\frac{d}{dt}E_k(\tilde{z})(t) \leq C(E_k(t))^p$$

for $k \geq 3$.

Proof. We will prove for $k = 3$ the rest of the cases are analog. Since $\|Q^2\|_{H^k} \leq C(E_k(t))^p$ and for the estimations (2.16) and (2.20), we get using (3.3)

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\partial_\alpha \tilde{z}\|^2_{L^2}(t) = \int_{\mathbb{T}} \partial^3_\alpha \tilde{z}(\alpha,t)\cdot Q^2(\alpha,t)\partial^3_\alpha \tilde{B}(\alpha,t)d\alpha \\
&+3\int_{\mathbb{T}} \partial^3_\alpha \tilde{z}(\alpha,t)\cdot \partial_\alpha Q^2(\alpha,t)\partial^2_\alpha \tilde{B}(\alpha,t)d\alpha \\
&+3\int_{\mathbb{T}} \partial^3_\alpha \tilde{z}(\alpha,t)\cdot \partial^2_\alpha Q^2(\alpha,t)\partial_\alpha \tilde{B}(\alpha,t)d\alpha \\
&+\int_{\mathbb{T}} \partial^3_\alpha \tilde{z}(\alpha,t)\cdot \partial^3_\alpha Q^2(\alpha,t)\tilde{B}(\alpha,t)d\alpha \\
&+\int_{\mathbb{T}} \partial^3_\alpha \tilde{z}(\alpha,t)\cdot \partial^3_\alpha(\tilde{c}(\alpha,t)\partial_\alpha \tilde{z}(\alpha,t))d\alpha \\
&\equiv I_1 + I_2 + I_3 + I_4 + I_5
\end{aligned}$$

where $\tilde{B}(\alpha,t) = BR(\tilde{\varpi}_1,\tilde{z})_{\tilde{z}}(\alpha,t) + BR(\tilde{\varpi}_2,\tilde{h})_{\tilde{z}}(\alpha,t)$. It is easy to see that

$$\begin{aligned}
&|I_2| \leq C\|\partial^3_\alpha \tilde{z}\|_{L^2}\|\partial_\alpha Q^2\|_{L^\infty}\|\partial^2_\alpha \tilde{B}\|_{L^2} \leq C(E_k(t))^p, \\
&|I_3| \leq C\|\partial^3_\alpha \tilde{z}\|_{L^2}\|Q^2\|_{H^2}\|\partial_\alpha \tilde{B}\|_{L^\infty} \leq C(E_k(t))^p, |I_4| \leq C\|\partial^3_\alpha \tilde{z}\|_{L^2}\|\partial^3_\alpha Q^2\|_{L^2}\|\tilde{B}\|_{L^\infty}.
\end{aligned}$$

We split I_1 in the following way

$$\begin{aligned}
I_1 =& \int_{\mathbb{T}} \partial^3_\alpha \tilde{z}(\alpha,t)\cdot Q^2(\alpha,t)\partial^3_\alpha BR(\tilde{\varpi}_1,\tilde{z})_{\tilde{z}}(\alpha,t)d\alpha \\
&+\int_{\mathbb{T}} \partial^3_\alpha \tilde{z}(\alpha,t)\cdot Q^2\partial^3_\alpha BR(\tilde{\varpi}_2,\tilde{h})_{\tilde{z}}(\alpha,t)d\alpha \equiv I_1^1 + I_1^2.
\end{aligned}$$

The most singular terms in I_1^2 are:

$$\begin{aligned}
I_1^{21} &= \frac{1}{2\pi}\int_{\mathbb{T}}\int_{\mathbb{R}} \partial^3_\alpha \tilde{z}(\alpha)\cdot Q^2(\alpha)\frac{(\partial^3_\alpha \tilde{z}(\alpha) - \partial^3_\alpha \tilde{h}(\alpha-\beta))^\perp}{|\tilde{z}(\alpha)-\tilde{h}(\alpha-\beta)|^2}\tilde{\varpi}_2(\alpha-\beta)d\beta d\alpha, \\
I_1^{22} &= -\frac{1}{\pi}\int_{\mathbb{T}}\int_{\mathbb{R}} \partial^3_\alpha \tilde{z}(\alpha)\cdot Q^2(\alpha)\frac{\Delta\tilde{z}\tilde{h}^\perp(\Delta\tilde{z}\tilde{h}\cdot\Delta\partial^3_\alpha \tilde{z}\tilde{h})}{|\tilde{z}(\alpha)-\tilde{h}(\alpha-\beta)|^4}\tilde{\varpi}_2(\alpha-\beta)d\beta d\alpha, \\
I_1^{23} &= \frac{1}{\pi}\int_{\mathbb{T}}\int_{\mathbb{R}} \partial^3_\alpha \tilde{z}(\alpha)\cdot Q^2(\alpha)\frac{(\tilde{z}(\alpha)-\tilde{h}(\alpha-\beta))^\perp}{|\tilde{z}(\alpha)-\tilde{h}(\alpha-\beta)|^2}\partial^3_\alpha\tilde{\varpi}_2(\alpha-\beta)d\beta d\alpha.
\end{aligned}$$

Since $\partial^3_\alpha \tilde{z}(\alpha)\cdot\partial^3_\alpha \tilde{z}(\alpha)^\perp = 0$,

$$\begin{aligned}
&I_1^{21} \leq C\|d(\tilde{z},\tilde{h})\|_{L^\infty}\|\partial^3_\alpha \tilde{z}\|_{L^2}\|Q^2\|_{L^\infty}\|\partial^3_\alpha \tilde{h}\|_{L^2}\|\tilde{\varpi}_2\|_{L^\infty}, \\
&I_1^{22} \leq C\|\partial^3_\alpha \tilde{z}\|_{L^2}\|Q^2\|_{L^\infty}\|d(\tilde{z},\tilde{h})\|_{L^\infty}(\|\partial^3_\alpha \tilde{z}\|_{L^2} + \|\partial^3_\alpha \tilde{h}\|_{L^2})\|\tilde{\varpi}_2\|_{L^\infty}.
\end{aligned}$$

For I_1^{23} using integrations by parts

$$
\begin{aligned}
I_1^{23} &= -\frac{1}{2\pi}\int_{\mathbb{T}}\int_{\mathbb{R}} \partial_\alpha^3\tilde{z}(\alpha)\cdot Q^2(\alpha)\frac{\partial_\alpha^\perp \tilde{h}(\alpha-\beta)}{|\tilde{z}(\alpha)-\tilde{h}(\alpha-\beta)|^2}\partial_\alpha^2\tilde{\varpi}_2(\alpha-\beta)d\beta d\alpha \\
&+ \frac{1}{\pi}\int_{\mathbb{T}}\int_{\mathbb{R}} \partial_\alpha^3\tilde{z}(\alpha)\cdot Q^2(\alpha)\frac{(\Delta\tilde{z}\tilde{h})^\perp \Delta\tilde{z}\tilde{h}\cdot\partial_\alpha\tilde{h}(\alpha-\beta)}{|\tilde{z}(\alpha)-\tilde{h}(\alpha-\beta)|^4}\partial_\alpha^2\tilde{\varpi}_2(\alpha-\beta)d\beta d\alpha \\
&\equiv I_1^{231} + I_1^{232}.
\end{aligned}
$$

Then, is clear that

$$
\begin{aligned}
I_1^{231} &\leq C\|\partial_\alpha^3\tilde{z}\|_{L^2}\|Q^2\|_{L^\infty}\|d(\tilde{z},\tilde{h})\|_{L^\infty}\|\partial_\alpha\tilde{h}\|_{L^\infty}\|\partial_\alpha^2\tilde{\varpi}_2\|_{L^2}, \\
I_1^{232} &\leq C\|\partial_\alpha^3\tilde{z}\|_{L^2}\|Q^2\|_{L^\infty}\|d(\tilde{z},\tilde{h})\|_{L^\infty}\|\partial_\alpha\tilde{h}\|_{L^\infty}\|\partial_\alpha^2\tilde{\varpi}_2\|_{L^2}.
\end{aligned}
$$

Then,

$$
I_1^2 \leq C(E_k(t))^p.
$$

For I_1^1 we write

$$
\begin{aligned}
I_1^1 &\leq C(E_k(t))^p + \frac{1}{2\pi}\int_{\mathbb{T}}\int_{\mathbb{R}} \partial_\alpha^3\tilde{z}(\alpha)\cdot Q^2\frac{(\tilde{z}(\alpha)-\tilde{z}(\alpha-\beta))^\perp}{|\tilde{z}(\alpha)-\tilde{z}(\alpha-\beta)|^2}\partial_\alpha^3\tilde{\varpi}_1(\alpha-\beta)d\beta d\alpha \\
&\leq C(E_k(t))^p + \frac{1}{2\pi}\int_{\mathbb{T}}\int_{\mathbb{R}} \partial_\alpha^3\tilde{z}(\alpha)\cdot Q^2\frac{\partial_\alpha^\perp\tilde{z}(\alpha)}{\beta|\partial_\alpha\tilde{z}(\alpha)|^2}\partial_\alpha^3\tilde{\varpi}_1(\alpha-\beta)d\beta d\alpha \equiv I_1^{11}.
\end{aligned}
$$

Using the definition of the Hilbert transform, we can write:

$$
I_1^{11} = \frac{1}{2}\int_{\mathbb{T}} \partial_\alpha^3\tilde{z}(\alpha)\cdot Q^2\frac{\partial_\alpha^\perp\tilde{z}(\alpha)}{|\partial_\alpha\tilde{z}(\alpha)|^2}H(\partial_\alpha^3\tilde{\varpi}_1)(\alpha)d\alpha.
$$

Since, $H(\partial_\alpha) = \Lambda$,

$$
\begin{aligned}
I_1^{11} &= \frac{1}{2}\int_{\mathbb{T}} \partial_\alpha^3\tilde{z}(\alpha)\cdot Q^2\frac{\partial_\alpha^\perp\tilde{z}(\alpha)}{|\partial_\alpha\tilde{z}(\alpha)|^2}\Lambda(\partial_\alpha^2\tilde{\varpi}_1)(\alpha)d\alpha \\
&= \frac{1}{2}\int_{\mathbb{T}} \Lambda(\partial_\alpha^3\tilde{z}\cdot Q^2\frac{\partial_\alpha^\perp\tilde{z}}{|\partial_\alpha\tilde{z}|^2})(\alpha)\partial_\alpha^2\tilde{\varpi}_1(\alpha)d\alpha.
\end{aligned}
$$

Now, we have that $\tilde{\varpi}_1(\alpha) = -2\tilde{B}(\alpha,t)\cdot\partial_\alpha\tilde{z}(\alpha) - 2\frac{\kappa^1\rho^2}{\mu^2}\partial_\alpha(P_2^{-1}(\tilde{z}(\alpha)))$ then we can split $I_1^{11} = I_1^{12} + I_1^{13}$ where:

$$
\begin{aligned}
I_1^{12} &= -\frac{1}{\tilde{A}}\int_{\mathbb{T}} \Lambda(\partial_\alpha^3\tilde{z}\cdot\partial_\alpha^\perp\tilde{z}Q^2)(\alpha)\partial_\alpha^2(\tilde{B}(\alpha)\cdot\partial_\alpha\tilde{z}(\alpha))d\alpha, \\
I_1^{13} &= -\frac{\kappa^1\rho^2}{\mu^2\tilde{A}}\int_{\mathbb{T}} \Lambda(\partial_\alpha^3\tilde{z}\cdot\partial_\alpha^\perp\tilde{z}Q^2)(\alpha)\partial_\alpha^3(P_2^{-1}(\tilde{z}(\alpha)))d\alpha.
\end{aligned}
$$

In order to estimate this two terms, we have to proceed in the same way as in the study of the non-splat existence. Doing that we get:

$$
I_1^{12} \leq C(E_k(t))^p - \frac{1}{\tilde{A}(t)}\int_{\mathbb{T}} Q^2(\alpha)\tilde{B}(\alpha)\cdot\partial_\alpha^\perp\tilde{z}(\alpha)\partial_\alpha^3\tilde{z}(\alpha)\cdot\Lambda(\partial_\alpha^3\tilde{z})(\alpha)d\alpha,
$$

$$I_1^{13} \leq C(E_k(t))^p - \frac{\kappa^1\rho^2}{\mu^2\tilde{A}}\int_{\mathbb{T}} \Lambda(\partial_\alpha^3\tilde{z}\cdot\partial_\alpha^{\perp}\tilde{z}Q^2)(\alpha)\nabla P_2^{-1}(\tilde{z}(\alpha))\cdot\partial_\alpha^3\tilde{z}(\alpha)d\alpha.$$

Spliting I_1^{13} in components:

$$I_1^{131} = \frac{\kappa^1\rho^2}{\mu^2\tilde{A}}\int_{\mathbb{T}} \Lambda(\partial_\alpha^3\tilde{z}_1\cdot\partial_\alpha^{\perp}\tilde{z}_2Q^2)(\alpha)\partial_{\tilde{x}_1}P_2^{-1}(\tilde{z}(\alpha))\cdot\partial_\alpha^3\tilde{z}_1(\alpha)d\alpha,$$
$$I_1^{132} = \frac{\kappa^1\rho^2}{\mu^2\tilde{A}}\int_{\mathbb{T}} \Lambda(\partial_\alpha^3\tilde{z}_1\cdot\partial_\alpha^{\perp}\tilde{z}_2Q^2)(\alpha)\partial_{\tilde{x}_2}P_2^{-1}(\tilde{z}(\alpha))\cdot\partial_\alpha^3\tilde{z}_2(\alpha)d\alpha,$$
$$I_1^{133} = -\frac{\kappa^1\rho^2}{\mu^2\tilde{A}}\int_{\mathbb{T}} \Lambda(\partial_\alpha^3\tilde{z}_2\cdot\partial_\alpha^{\perp}\tilde{z}_1Q^2)(\alpha)\partial_{\tilde{x}_1}P_2^{-1}(\tilde{z}(\alpha))\cdot\partial_\alpha^3\tilde{z}_1(\alpha)d\alpha,$$
$$I_1^{134} = -\frac{\kappa^1\rho^2}{\mu^2\tilde{A}}\int_{\mathbb{T}} \Lambda(\partial_\alpha^3\tilde{z}_2\cdot\partial_\alpha^{\perp}\tilde{z}_1Q^2)(\alpha)\partial_{\tilde{x}_2}P_2^{-1}(\tilde{z}(\alpha))\cdot\partial_\alpha^3\tilde{z}_2(\alpha)d\alpha.$$

Using,

$$\|\Lambda(fg)-g\Lambda(f)\|_{L^2} \leq C\|g\|_{\mathcal{C}^{1,\frac{1}{3}}}\|f\|_{L^2}$$

we have,

$$I_1^{131} \leq C(E_k(t))^p + \frac{\kappa^1\rho^2}{\mu^2\tilde{A}(t)}\int_{\mathbb{T}} Q^2(\alpha)\partial_{\tilde{x}_1}P_2^{-1}(\tilde{z}(\alpha))\partial_\alpha\tilde{z}_2(\alpha)\partial_\alpha^3\tilde{z}_1(\alpha)\Lambda(\partial_\alpha^3\tilde{z}_1)(\alpha)d\alpha,$$
$$I_1^{134} \leq C(E_k(t))^p + \frac{\kappa^1\rho^2}{\mu^2\tilde{A}(t)}\int_{\mathbb{T}} Q^2(\alpha)\partial_{\tilde{x}_2}P_2^{-1}(\tilde{z}(\alpha))\partial_\alpha\tilde{z}_1(\alpha)\partial_\alpha^3\tilde{z}_2(\alpha)\Lambda(\partial_\alpha^3\tilde{z}_2)(\alpha)d\alpha.$$

Moreover, taking into account

$$\partial_\alpha^3\tilde{z}_2(\alpha)\partial_\alpha\tilde{z}_2(\alpha) = -\partial_\alpha\tilde{z}_1(\alpha)\partial_\alpha^3\tilde{z}_1(\alpha) + |\partial_\alpha^2\tilde{z}(\alpha)|^2,$$

we have

$$I_1^{132} \leq C(E_k(t))^p + \frac{\kappa^1\rho^2}{\mu^2\tilde{A}(t)}\int_{\mathbb{T}} Q^2(\alpha)\partial_{\tilde{x}_2}P_2^{-1}(\tilde{z}(\alpha))\partial_\alpha\tilde{z}_1(\alpha)\partial_\alpha^3\tilde{z}_1(\alpha)\Lambda(\partial_\alpha^3\tilde{z}_1)(\alpha)d\alpha,$$
$$I_1^{133} \leq C(E_k(t))^p + \frac{\kappa^1\rho^2}{\mu^2\tilde{A}(t)}\int_{\mathbb{T}} Q^2(\alpha)\partial_{\tilde{x}_1}P_2^{-1}(\tilde{z}(\alpha))\partial_\alpha\tilde{z}_2(\alpha)\partial_\alpha^3\tilde{z}_2(\alpha)\Lambda(\partial_\alpha^3\tilde{z}_2)(\alpha)d\alpha.$$

Therefore, adding all terms we can check that

$$I_1^{13} \leq C(E_k(t))^p - \frac{\kappa^1\rho^2}{\mu^2\tilde{A}(t)}\int_{\mathbb{T}} Q^2(t)\nabla P_2^{-1}(\tilde{z}(\alpha))\cdot\partial_\alpha^{\perp}\tilde{z}(\alpha)\partial_\alpha^3\tilde{z}(\alpha)\cdot\Lambda(\partial_\alpha^3\tilde{z})(\alpha)d\alpha.$$

Thus,

$$I_1^{11} \leq C(E_k(t))^p - \frac{\mu^2}{\kappa^1\tilde{A}(t)}\int_{\mathbb{T}} Q^2(\alpha)\tilde{\sigma}(\alpha)\partial_\alpha^3\tilde{z}(\alpha)\cdot\Lambda(\partial_\alpha^3\tilde{z})(\alpha)d\alpha.$$

To finish the proof we only have to estimate the term I_5. It is clear that:

$$
\begin{aligned}
I_5 = &\int_{\mathbb{T}} \partial_\alpha^3 \tilde{z}(\alpha,t)\cdot\partial_\alpha^4\tilde{z}(\alpha,t)\tilde{c}(\alpha,t)d\alpha + 3\int_{\mathbb{T}}|\partial_\alpha^3\tilde{z}(\alpha,t)|^2\partial_\alpha\tilde{c}(\alpha,t)d\alpha\\
&+3\int_{\mathbb{T}}\partial_\alpha^3\tilde{z}(\alpha,t)\cdot\partial_\alpha^2\tilde{z}(\alpha,t)\partial_\alpha^2\tilde{c}(\alpha,t)d\alpha+\int_{\mathbb{T}}\partial_\alpha^3\tilde{z}(\alpha,t)\cdot\partial_\alpha\tilde{z}(\alpha,t)\partial_\alpha^3\tilde{c}(\alpha,t)d\alpha\\
&\equiv I_5^1+I_5^2+I_5^3+I_5^4.
\end{aligned}
$$

Since $I_5^2 = -6I_5^1$ and

$$I_5^1 = -\frac{1}{2}\int_{\mathbb{T}}|\partial_\alpha^3\tilde{z}(\alpha)|^2\partial_\alpha\tilde{c}(\alpha)d\alpha \le C\|\partial_\alpha\tilde{c}\|_{L^\infty}\|\partial_\alpha^3\tilde{z}\|_{L^2}^2,$$

we have

$$I_5^1 + I_5^2 \le C(E_k(t))^p.$$

Using the definition of $\tilde{c}$ we have:

$$\partial_\alpha^2\tilde{c}(\alpha) = -\frac{\partial_\alpha^2\tilde{z}(\alpha)}{\tilde{A}(t)}\cdot\partial_\alpha(Q^2\tilde{B}) - \frac{\partial_\alpha\tilde{z}(\alpha)}{\tilde{A}(t)}\cdot\partial_\alpha^2(Q^2\tilde{B}).$$

Recall that $\tilde{B} = BR(\tilde{\varpi}_1,\tilde{z})_{\tilde{z}} + BR(\tilde{\varpi}_2,\tilde{h})_{\tilde{z}}$. Then,

$$
\begin{aligned}
|I_5^3| \le\, & C\|\mathcal{F}(\tilde{z})\|_{L^\infty}\|z\|_{C^2}^2(\|\partial_\alpha Q^2\|_{L^\infty}\|\tilde{B}\|_{L^2}+\|Q^2\|_{L^2}\|\partial_\alpha\tilde{B}\|_{L^\infty})\|\partial_\alpha^3\tilde{z}\|_{L^2}\\
&+C\|\mathcal{F}(\tilde{z})\|_{L^\infty}^{\frac{1}{2}}\|z\|_{C^2}(\|\partial_\alpha^2Q^2\|_{L^2}\|\tilde{B}\|_{L^\infty}+2\|\partial_\alpha Q^2\|_{L^2}\|\partial_\alpha\tilde{B}\|_{L^\infty}+\|\tilde{c}\|_{L^\infty}\|\partial_\alpha^2\tilde{B}\|_{L^2})\|\partial_\alpha^3\tilde{z}\|_{L^2}.
\end{aligned}
$$

Using (2.20), $|I_5^3| \le C(E_k(t))^p$. Finally, if we derivate twice $\tilde{A}(t)$ we get $\partial_\alpha\tilde{z}\cdot\partial_\alpha^3\tilde{z} = -|\partial_\alpha^2\tilde{z}|^2$. Then, integrating by parts, $I_5^4 = \frac{2}{3}I_5^3$.

Following the proceeding with $m(Q^2\tilde{\sigma}) > 0$ and $2f\Lambda(f) - \Lambda(f^2) \ge 0$ we get the desired estimate, and furthermore the local-existence. ∎

3.2.3 Stability for the Inhomogeneous Muskat Problem

In this section we will study a stability result for the transformed problem.

Theorem 3.2.2. *Let $x(\alpha,t)$ and $y(\alpha,t)$ be two curves which satisfy the equations (3.3),(3.4),(3.5),(3.6) and (3.7). Then the following estimate holds:*

$$\frac{d}{dt}\|x-y\|_{H^1}(t) \le C(\sup_{[0,T]}E_3(x)(t)+\sup_{[0,t]}E_3(y)(t))^p\|x-y\|_{H^1}(t)$$

where C and p are universal constants.

Recall that

$$E_k(\tilde{z})(t) \equiv \|\tilde{z}\|_{H^k}^2 + \|\mathcal{F}(\tilde{z})\|_{L^\infty}^2 + \|d(\tilde{z},\tilde{h})\|_{L^\infty}^2 + \frac{1}{m(Q^2\tilde{\sigma})(t)} + \sum_{l=0}^{4}\frac{1}{m(q^l)(t)}.$$

Proof. Let us consider $\gamma_i(\alpha,t)$ for $i = 1,2$ the vorticity amplitudes for the curve $x(\alpha,t)$ and $\xi_i(\alpha,t)$ for $i = 1,2$ the vorticity amplitudes for $y(\alpha,t)$. We denote for Q_x^2, Q_y^2 the function

$Q^2_\cdot = |\frac{dP}{dw}(P^{-1}(\cdot))|^2$ and c_x, c_y the parametrization constants associated to x and y respectively.

We want to estimate the L^2-norm for the function $z(\alpha,t) = x(\alpha,t) - y(\alpha,t)$:

$$\frac{1}{2}\frac{d}{dt}\|z\|^2_{L^2} = \int_{\mathbb{T}} z \cdot z_t d\alpha \equiv I_1 + I_2 + I_3 + I_4$$

where

$$\begin{aligned}
I_1 &= \int_{\mathbb{T}} z \cdot (Q^2_x - Q^2_y)(BR(\gamma_1, x)_x + BR(\gamma_2, h)_x) d\alpha,\\
I_2 &= \int_{\mathbb{T}} z \cdot Q^2_y(BR(\gamma_1, x)_x + BR(\gamma_2, h)_x - BR(\xi_1, y)_y - BR(\xi_2, h)_y) d\alpha,\\
I_3 &= \int_{\mathbb{T}} z \cdot (c_x - c_y)\partial_\alpha x d\alpha,\\
I_4 &= \int_{\mathbb{T}} z \cdot c_y \partial_\alpha z d\alpha.
\end{aligned}$$

During time of existence $T > 0$ we have $\sup E_3(x)$ and $\sup E_3(y)$ bounded so that we write

$$C(\sup_{[0,T]} E_3(x)(t) + \sup_{[0,t]} E_3(y)(t))^p \leq C$$

by abused notation. For I_1 and I_4 directly

$$\begin{aligned}
I_1 &\leq \|z\|_{L^\infty}\|Q^2_x - Q^2_y\|_{L^2}(\|BR(\gamma_1, x)_x\|_{L^2} + \|BR(\gamma_2, h)_x\|_{L^2}) \leq C\|z\|^2_{H^1},\\
I_4 &\leq C\|L^\infty\|\|c_y\|_{L^2}\|\partial_\alpha z\|_{L^2} \leq C\|z\|^2_{H^1}.
\end{aligned}$$

We can split $I_2 = I_2^1 + I_2^2$ where

$$\begin{aligned}
I_2^1 &= \int_{\mathbb{T}} z \cdot Q^2_y(BR(\gamma_1, x)_x - BR(\xi_1, y)_y) d\alpha,\\
I_2^2 &= \int_{\mathbb{T}} z \cdot Q^2_y(BR(\gamma_2, h)_x - BR(\xi_2, h)_y) d\alpha.
\end{aligned}$$

We split I_2^1 in the following terms:

$$\begin{aligned}
I_2^{11} &= \frac{1}{2\pi}\int_{\mathbb{T}} z \cdot Q^2_y \int_{\mathbb{R}} \frac{(\Delta z)^\perp}{|\Delta x|^2}\gamma_1(\alpha - \beta) d\beta d\alpha,\\
I_2^{12} &= \frac{1}{2\pi}\int_{\mathbb{T}} z \cdot Q^2_y \int_{\mathbb{R}} (\Delta y)^\perp (\frac{1}{|\Delta x|^2} - \frac{1}{|\Delta y|^2})\gamma_1(\alpha - \beta) d\beta d\alpha,\\
I_2^{13} &= \frac{1}{2\pi}\int_{\mathbb{T}} z \cdot Q^2_y \int_{\mathbb{R}} \frac{(\Delta y)^\perp}{|\Delta y|^2} w_1(\alpha - \beta) d\beta d\alpha.
\end{aligned}$$

where $w_1 = \gamma_1 - \xi_1$ and $\Delta f = f(\alpha) - f(\alpha - \beta)$.

As always can estimate I_2^{11} as follows:

$$I_2^{11} = \frac{1}{2\pi}\int_{\mathbb{T}}\int_{\mathbb{R}} z \cdot Q^2_y (\frac{(\Delta z)^\perp}{|\Delta x|^2} - \frac{\partial_\alpha^\perp z(\alpha)}{\beta|\partial_\alpha x(\alpha)|^2})\gamma_1(\alpha - \beta) d\beta d\alpha$$

$$+\frac{1}{2\pi}\int_{\mathbb{T}} z\cdot Q_y^2\frac{\partial_\alpha^\perp z(\alpha)}{|\partial_\alpha x(\alpha)|^2}H(\gamma_1)(\alpha)d\alpha \leq C\|z\|_{H^1}^2.$$

The term I_2^{12} is:

$$I_2^{12}=\frac{1}{2\pi}\int_{\mathbb{T}}\int_{\mathbb{R}} z\cdot Q_y^2(\Delta y)^\perp\frac{\Delta z\cdot(\Delta y+\Delta x)}{|\Delta x|^2|\Delta y|^2}\gamma_1(\alpha-\beta)d\beta d\alpha.$$

We do the same technique as before twice, for x and for y.

First:

$$\begin{aligned}I_2^{12}&=\frac{1}{2\pi}\int_{\mathbb{T}}\int_{\mathbb{R}} z\cdot Q_y^2(\Delta y)^\perp\frac{\Delta z\cdot(\Delta y+\Delta x)}{|\Delta y|^2}\Big(\frac{1}{|\Delta x|^2}-\frac{1}{\beta^2|\partial_\alpha x(\alpha)|^2}\Big)\gamma_1(\alpha-\beta)d\beta d\alpha\\ &+\frac{1}{2\pi}\int_{\mathbb{T}}\int_{\mathbb{R}} z\cdot Q_y^2(\Delta y)^\perp\frac{\Delta z\cdot(\Delta y+\Delta x)}{\beta^2|\partial_\alpha x(\alpha)|^2|\Delta y|^2}\gamma_1(\alpha-\beta)d\beta d\alpha\equiv I_2^{121}+I_2^{122}.\end{aligned}$$

Since,

$$\Delta f=\beta\int_0^1\partial_\alpha f ds$$

and

$$\frac{1}{|\Delta x|^2}-\frac{1}{\beta^2|\partial_\alpha x|^2}=\frac{\beta\int_0^1\int_0^1\partial_\alpha^2 x dsdt\cdot\int_0^1(\partial_\alpha x+\partial_\alpha x)ds}{|\Delta x|^2|\partial_\alpha x|^2}$$

then $I_2^{121}\leq C\|z\|_{H^1}^2$. On the other hand,

$$\begin{aligned}I_2^{122}&=\frac{1}{2\pi}\int_{\mathbb{T}}\int_{\mathbb{R}} z\cdot Q_y^2(\Delta y)^\perp\frac{\Delta z\cdot(\Delta y+\Delta x)}{\beta^2|\partial_\alpha x(\alpha)|^2}\Big(\frac{1}{|\Delta y|^2}-\frac{1}{\beta^2|\partial_\alpha y(\alpha)|^2}\Big)\gamma_1(\alpha-\beta)d\beta d\alpha\\ &+\frac{1}{2\pi}\int_{\mathbb{T}}\int_{\mathbb{R}} z\cdot Q_y^2(\Delta y)^\perp\frac{\Delta z\cdot(\Delta y+\Delta x)}{\beta^4|\partial_\alpha x(\alpha)|^2|\partial_\alpha y(\alpha)|^2}\gamma_1(\alpha-\beta)d\beta d\alpha\\ &\leq C\|z\|_{H^1}^2+\frac{1}{2\pi}\int_{\mathbb{T}} z\cdot Q_y^2\partial_\alpha^\perp y(\alpha)\frac{\partial_\alpha z(\alpha)\cdot(\partial_\alpha y(\alpha)+\partial_\alpha x(\alpha))}{|\partial_\alpha x(\alpha)|^2|\partial_\alpha y(\alpha)|^2}H(\gamma_1)(\alpha)d\alpha\\ &\leq C\|z\|_{H^1}^2.\end{aligned}$$

And for I_2^{13},

$$\begin{aligned}I_2^{13}&=\frac{1}{2\pi}\int_{\mathbb{T}}\int_{\mathbb{R}} z\cdot Q_y^2\Big(\frac{(\Delta y)^\perp}{|\Delta y|^2}-\frac{\partial_\alpha^\perp y(\alpha)}{\beta|\partial_\alpha y(\alpha)|^2}\Big)w_1(\alpha-\beta)d\beta d\alpha\\ &+\frac{1}{2\pi}\int_{\mathbb{T}} z\cdot Q_y^2\frac{\partial_\alpha^\perp y(\alpha)}{|\partial_\alpha y(\alpha)|^2}H(w_1)(\alpha)d\alpha\leq C\|z\|_{L^2}\|w_1\|_{L^2}.\end{aligned}$$

Therefore, we only have to deal with $\|w_1\|_{L^2}$. To do that, we consider:

$$\begin{aligned}&w_1+2BR(w_1,x)_x\cdot\partial_\alpha x+2BR(w_2,h)_x\cdot\partial_\alpha x\\ &=2BR(\xi_1,y)_y\cdot\partial_\alpha^y-2BR(\xi_1,x)_x\cdot\partial_\alpha x+2BR(\xi_2,h)_y\cdot\partial_\alpha y-2BR(\xi_2,h)_x\cdot\partial_\alpha x\\ &+2\kappa^1\frac{\rho^2}{\mu^2}(\nabla P_2^{-1}(y)\cdot\partial_\alpha y-\nabla P_2^{-1}(x)\cdot\partial_\alpha x)\end{aligned}$$

and, for $K = \frac{\kappa^2-\kappa^1}{\kappa^1+\kappa^2}$,

$$\begin{aligned}&w_2 + 2KBR(w_1,x)_h \cdot \partial_\alpha h + 2KBR(w_2,h)_h \cdot \partial_\alpha h\\ &= -2KBR(\xi_1,x)_h \cdot \partial_\alpha h + 2KBR(\xi_1,y)_h \cdot \partial_\alpha h.\end{aligned}$$

If we consider $w = (w_1 \quad w_2)^t$ then we have

$$w + \begin{pmatrix} 1 & 0 \\ 0 & K \end{pmatrix} \mathcal{T}(w) = \begin{pmatrix} f \\ g \end{pmatrix}$$

where $\mathcal{T}$ is the operator defined in (2.1) and

$$\begin{aligned}&f = 2BR(\xi_1,y)_y \cdot \partial_\alpha^y - 2BR(\xi_1,x)_x \cdot \partial_\alpha x + 2BR(\xi_2,h)_y \cdot \partial_\alpha y - 2BR(\xi_2,h)_x \cdot \partial_\alpha x\\ &+ 2\kappa^1 \frac{\rho^2}{\mu^2}(\nabla P_2^{-1}(y) \cdot \partial_\alpha y - \nabla P_2^{-1}(x) \cdot \partial_\alpha x),\\ &g = -2KBR(\xi_1,x)_h \cdot \partial_\alpha h + 2KBR(\xi_1,y)_h \cdot \partial_\alpha h.\end{aligned}$$

In chapter 2, we have proved that the $\mathcal{T}$ operator have inverse and that inverse is bounded by $E_3(x)(t)$. Therefore, this allow us to estimate:

$$\|w\|_{L^2} \leq C\|(fg)^t\|_{L^2}$$

For $\|(fg)^t\|_{L^2}$ we study $\|f\|_{L^2}$ and $\|g\|_{L^2}$ separately. Let us study first the L^2-norm off f:

$$\begin{aligned}&\|f\|_{L^2} = \|2BR(\xi_1,y)_y \cdot \partial_\alpha y + 2BR(\gamma_1,x)_x \cdot \partial_\alpha x + 2\kappa^1 \frac{\rho^2}{\mu^2}(\nabla P_2^{-1}(y) \cdot \partial_\alpha y - \nabla P_2^{-1}(x) \cdot \partial_\alpha x)\|_{L^2}\\ &+ \|2BR(\xi_2,h)_x \cdot \partial_\alpha x + 2BR(\xi_2,h)_y \cdot \partial_\alpha y\|_{L^2} \equiv \|f_1\|_{L^2} + \|f_2\|_{L^2}\end{aligned}$$

The term $\|f_1\|_{L^2} \leq C\|z\|_{H^1}^2$, for more details see [3].

For f_2,

$$\begin{aligned}f_2 &= \frac{1}{\pi}\int_{\mathbb{R}} \frac{z^\perp(\alpha) \cdot \partial_\alpha x(\alpha)}{|x(\alpha) - h(\alpha-\beta)|^2}\xi_2(\alpha-\beta)d\beta\\ &+ \frac{1}{\pi}\int_{\mathbb{R}} (y(\alpha) - h(\alpha-\beta))^\perp \cdot (\frac{\partial_\alpha x(\alpha)}{|x(\alpha) - h(\alpha-\beta)|^2} - \frac{\partial_\alpha y(\alpha)}{|y(\alpha) - h(\alpha-\beta)|^2})\xi_2(\alpha-\beta)d\beta\\ &\equiv f_2^1 + f_2^2.\end{aligned}$$

Directly,

$$\|f_2^1\|_{L^2} \leq C\|d(x,h)\|_{L^\infty}\|\partial_\alpha x\|_{L^\infty}\|\xi_2\|_{L^2}\|z\|_{L^2}.$$

For f_2^2,

$$\begin{aligned}f_2^2 &= \frac{1}{\pi}\int_{\mathbb{R}} (y(\alpha) - h(\alpha-\beta))^\perp \cdot \frac{\partial_\alpha z(\alpha)}{|x(\alpha) - h(\alpha-\beta)|^2}\xi_2(\alpha-\beta)d\beta\\ &+ \frac{1}{\pi}\int_{\mathbb{R}} (y(\alpha) - h(\alpha-\beta))^\perp \cdot \partial_\alpha y(\alpha)(\frac{|y(\alpha)|^2 - |x(\alpha) + 2z(\alpha) \cdot h(\alpha-\beta)|^2}{|x(\alpha) - h(\alpha-\beta)|^2|y(\alpha) - h(\alpha-\beta)|^2})\xi_2(\alpha-\beta)d\beta\\ &\equiv f_2^{21} + f_2^{22}\end{aligned}$$

then

$$
\begin{aligned}
&\|f_2^{21}\|_{L^2} \leq C\|d(x,h)\|_{L^\infty}(\|y\|_{L^\infty} + \|h\|_{L^\infty})\|\partial_\alpha z\|_{L^2}\|\xi_2\|_{L^2} \leq C\|z\|_{H^1},\\
&\|f_2^{22}\|_{L^2} \leq C(\|y\|_{L^\infty} + \|h\|_{L^\infty})\|\partial_\alpha y\|_{L^\infty}\|d(x,h)\|_{L^\infty}\|d(y,h)\|_{L^\infty}(\|y\|_{L^2}\\
&+ \|x\|_{L^2} + \|z\|_{L^2}\|h\|_{L^\infty})\|\xi_2\|_{L^2} \leq C\|z\|_{H^1}.
\end{aligned}
$$

Therefore,

$$\|f\|_{L^2} \leq C\|z\|_{H^1}.$$

Finally,

$$
\begin{aligned}
g &= \frac{1}{\pi}\int_{\mathbb{R}} \frac{z^\perp(\alpha-\beta)\cdot\partial_\alpha h(\alpha)}{|h(\alpha)-y(\alpha-\beta)|^2}\xi_1(\alpha-\beta)d\beta\\
&+ \frac{1}{\pi}\int_{\mathbb{R}} \frac{(h(\alpha)-x(\alpha-\beta))^\perp\cdot\partial_\alpha h(\alpha)(|x(\alpha)|^2-|y(\alpha)|^2)}{|h(\alpha)-y(\alpha-\beta)|^2|h(\alpha)-x(\alpha-\beta)|^2}d\beta\\
&+ \frac{2}{\pi}\int_{\mathbb{R}} \frac{(h(\alpha)-x(\alpha-\beta))^\perp\cdot\partial_\alpha h(\alpha)z(\alpha)\cdot h(\alpha)}{|h(\alpha)-y(\alpha-\beta)|^2|h(\alpha)-x(\alpha-\beta)|^2}d\beta\\
&\leq C\|z\|_{H^1}.
\end{aligned}
$$

Thus, $\|w\|_{L^2} \leq C\|z\|_{H^1}^2$ and therefore, $I_2^1 \leq C\|z\|_{H^1}^2$.

Let us study the term I_2^2. We can split it in the following terms:

$$
\begin{aligned}
I_2^{21} &= \int_{\mathbb{T}}\int_{\mathbb{R}} z(\alpha)\cdot Q_y^2(\alpha)\frac{z^\perp(\alpha)\cdot\gamma_2(\alpha-\beta)}{|x(\alpha)-h(\alpha-\beta)|^2}d\beta d\alpha,\\
I_2^{22} &= \int_{\mathbb{T}}\int_{\mathbb{R}} z(\alpha)\cdot Q_y^2(\alpha)(\Delta yh)^\perp(\frac{1}{|\Delta xh|^2}-\frac{1}{|\Delta yh|^2})\gamma_2(\alpha-\beta)d\beta d\alpha,\\
I_2^{23} &= \int_{\mathbb{T}}\int_{\mathbb{R}} z(\alpha)\cdot Q_y^2(\alpha)\frac{(y(\alpha)-h(\alpha-\beta))^\perp}{|y(\alpha)-h(\alpha-\beta)|^2}w_2(\alpha-\beta)d\beta d\alpha.
\end{aligned}
$$

Then,

$$
\begin{aligned}
&I_2^{21} \leq C\|d(x,h)\|_{L^\infty}\|Q_y^2\|_{L^\infty}\|z\|_{L^2}^2\|\gamma_2\|_{L^\infty} \leq C\|z\|_{H^1}^2,\\
&I_2^{22} = -\int_{\mathbb{T}}\int_{\mathbb{R}} z(\alpha)\cdot Q_y^2(\alpha)(\Delta yh)^\perp(\frac{z(\alpha)\cdot(y(\alpha)+x(\alpha)-2h(\alpha-\beta))}{|\Delta xh|^2|\Delta yh|^2})\gamma_2(\alpha-\beta)d\beta d\alpha\\
&\leq C\|d(x,h)\|_{L^\infty}^{\frac{1}{2}}\|Q_y^2\|_{L^\infty}\|z\|_{L^2}^2\|\gamma_2\|_{L^\infty} \leq C\|z\|_{H^1}^2,\\
&I_2^{23} \leq C\|d(y,h)\|_{L^\infty}^{\frac{1}{2}}\|Q_y^2\|_{L^\infty}\|z\|_{L^2}\|w_2\|_{L^2} \leq C\|z\|_{H^1}^2.
\end{aligned}
$$

Now we have to estimate the term I_3. For that we consider $c_x - c_y = G_1 + G_2$:

$$
\begin{aligned}
G_1 &= \frac{\alpha+\pi}{2\pi}\int_{\mathbb{T}} \frac{\partial_\beta x(\beta,t)}{|\partial_\beta x(\beta,t)|^2}\cdot\partial_\beta(Q_x^2(BR(\gamma_1,x)_x + BR(\gamma_2,h)_x))(\beta,t)\\
&- \frac{\partial_\beta y(\beta,t)}{|\partial_\beta y(\beta,t)|^2}\cdot\partial_\beta(Q_y^2(BR(\xi_1,y)_y + BR(\xi_2,h)_y))(\beta,t)d\beta,\\
G_2 &= -\int_{-\pi}^{\alpha} \frac{\partial_\beta x(\beta,t)}{|\partial_\beta x(\beta,t)|^2}\cdot\partial_\beta(Q_x^2(BR(\gamma_1,x)_x + BR(\gamma_2,h)_x))(\beta,t)\\
&- \frac{\partial_\beta y(\beta,t)}{|\partial_\beta y(\beta,t)|^2}\cdot\partial_\beta(Q_y^2(BR(\xi_1,y)_y + BR(\xi_2,h)_y))(\beta,t)d\beta.
\end{aligned}
$$

In order to estimate G_1 we split it as follows:

$$
\begin{aligned}
G_1^1 = {} & \frac{\alpha+\pi}{2\pi}\int_{\mathbb{T}} \frac{\partial_\beta x(\beta,t)}{|\partial_\beta x(\beta,t)|^2}\cdot\partial_\beta(Q_x^2 BR(\gamma_1,x)_x)(\beta,t)\\
& -\frac{\partial_\beta y(\beta,t)}{|\partial_\beta y(\beta,t)|^2}\cdot\partial_\beta(Q_y^2 BR(\xi_1,y)_y)(\beta,t)d\beta;\\
G_1^2 = {} & \frac{\alpha+\pi}{2\pi}\int_{\mathbb{T}} \frac{\partial_\beta x(\beta,t)}{|\partial_\beta x(\beta,t)|^2}\cdot\partial_\beta(Q_x^2 BR(\gamma_2,h)_x)(\beta,t)\\
& -\frac{\partial_\beta y(\beta,t)}{|\partial_\beta y(\beta,t)|^2}\cdot\partial_\beta(Q_y^2 BR(\xi_2,h)_y)(\beta,t)d\beta.
\end{aligned}
$$

Let us make the following break:

$$
\begin{aligned}
G_{1,1}^1 &= \frac{\alpha+\pi}{2\pi}\int_{\mathbb{T}} \frac{\partial_\beta x(\beta,t)}{|\partial_\beta x(\beta,t)|^2}\cdot\partial_\beta((Q_x^2-Q_y^2)BR(\gamma_1,x)_x)(\beta,t)d\beta;\\
G_{1,2}^1 &= \frac{\alpha+\pi}{2\pi}\int_{\mathbb{T}} \frac{\partial_\beta x(\beta,t)}{|\partial_\beta x(\beta,t)|^2}\cdot\partial_\beta Q_y^2(\beta)(BR(\gamma_1,x)_x-BR(\xi_1,y)_y)(\beta,t)d\beta;\\
G_{1,3}^1 &= \frac{\alpha+\pi}{2\pi}\int_{\mathbb{T}} \frac{\partial_\beta x(\beta,t)}{|\partial_\beta x(\beta,t)|^2}\cdot Q_y^2(\beta)\partial_\beta(BR(\gamma_1,x)_x-BR(\xi_1,y)_y)(\beta,t)d\beta;\\
G_{1,4}^1 &= \frac{\alpha+\pi}{2\pi}\int_{\mathbb{T}} \frac{\partial_\beta z(\beta,t)}{|\partial_\beta x(\beta,t)|^2}\cdot\partial_\beta(Q_y^2 BR(\xi_1,y)_y)(\beta,t)d\beta;\\
G_{1,5}^1 &= \frac{\alpha+\pi}{2\pi}\int_{\mathbb{T}} \partial_\beta(Q_y^2 BR(\xi_1,y)_y)(\beta,t)\cdot\partial_\alpha y(\beta)(\frac{1}{|\partial_\alpha x(\beta)|^2}-\frac{1}{|\partial_\alpha y(\beta)|^2})d\beta.
\end{aligned}
$$

We can proceed as before to get

$$
|G_{1,1}^1|+|G_{1,2}^1|+|G_{1,4}^1|+|G_{1,5}^1|\le C\|z\|_{H^1}.
$$

The term $G_{1,3}^1$ is more delicated, then we will split further: $G_{1,3}^1=G_{1,31}^1+G_{1,32}^1+G_{1,33}^1+G_{1,34}^1+G_{1,35}^1+G_{1,36}^1$, where

$$
\begin{aligned}
G_{1,31}^1 &= \frac{1}{2\pi}\int_{\mathbb{T}}\int_{\mathbb{R}} Q_y^2(\beta)\frac{(\Delta x)^\perp}{|\Delta x|^2}\cdot\frac{\partial_\beta x(\beta)}{|\partial_\beta x(\beta)|^2}\partial_\beta w_1(\beta-\tau)d\tau d\beta,\\
G_{1,32}^1 &= \frac{1}{2\pi}\int_{\mathbb{T}}\int_{\mathbb{R}} Q_y^2(\beta)(\frac{(\Delta x)^\perp}{|\Delta x|^2}-\frac{(\Delta y)^\perp}{|\Delta y|^2})\cdot\frac{\partial_\beta x(\beta)}{|\partial_\beta x(\beta)|^2}\partial_\beta \xi_1(\beta-\tau)d\tau d\beta,\\
G_{1,33}^1 &= \frac{1}{2\pi}\int_{\mathbb{T}}\int_{\mathbb{R}} Q_y^2(\beta)\frac{\Delta\partial_\beta^\perp z(\beta)}{|\Delta x|^2}\cdot\frac{\partial_\beta x(\beta)}{|\partial_\beta x(\beta)|^2}\gamma_1(\beta-\tau)d\tau d\beta,\\
G_{1,34}^1 &= \frac{1}{2\pi}\int_{\mathbb{T}}\int_{\mathbb{R}} Q_y^2(\beta)(\Delta\partial_\beta y)^\perp(\frac{\gamma_1(\beta-\tau)}{|\Delta x|^2}-\frac{\xi_1(\beta-\tau)}{|\Delta y|^2})\cdot\frac{\partial_\beta x(\beta)}{|\partial_\beta x(\beta)|^2}d\tau d\beta,\\
G_{1,35}^1 &= -\frac{1}{\pi}\int_{\mathbb{T}}\int_{\mathbb{R}} Q_y^2(\beta)\frac{(\Delta x)^\perp}{|\Delta x|^4}\cdot\frac{\partial_\beta x(\beta)}{|\partial_\beta x(\beta)|^2}\Delta x\cdot\Delta\partial_\beta z(\beta)\gamma_1(\beta-\tau)d\tau d\beta,\\
G_{1,36}^1 &= -\frac{1}{\pi}\int_{\mathbb{T}}\int_{\mathbb{R}} \frac{Q_y^2\partial_\beta x(\beta)}{|\partial_\beta x(\beta)|^2}\cdot(\frac{(\Delta x)^\perp\Delta x\cdot\Delta\partial_\beta y}{|\Delta x|^4}\gamma_1(\beta-\tau)-\frac{(\Delta y)^\perp\Delta y\cdot\Delta\partial_\beta y}{|\Delta y|^4}\xi_1(\beta-\tau))d\tau d\beta.
\end{aligned}
$$

The less singular terms can be controlled as before.

$$
|G_{1,32}^1|+|G_{1,34}^1|+|G_{1,36}^1|\le C\|z\|_{H^1}.
$$

Let us study the most singular terms. Since $\partial_\beta^\perp x\cdot\partial_\beta x=0$ we can write $G^1_{1,31}$ as follows:

$$G^1_{1,31}=\frac{1}{2\pi}\int_{\mathbb{T}}\int_{\mathbb{R}}Q^2_y(\beta)\frac{(\Delta x)^\perp-\tau\partial_\beta^\perp x(\beta)}{|\Delta x|^2}\cdot\frac{\partial_\beta x(\beta)}{|\partial_\beta x(\beta)|^2}\partial_\beta w_1(\beta-\tau)d\tau d\beta.$$

Then, using integration by parts we get $G^1_{1,31}\leq C\|w_1\|_{L^2}\leq C\|z\|_{H^1}$. For $G^1_{1,33}$ we integrate by parts:

$$G^1_{1,33}=-\frac{1}{2\pi}\int_{\mathbb{T}}\int_{\mathbb{R}}\Delta^\perp z(\beta)\cdot\partial_\beta\Big(\frac{Q^2_y(\beta)\partial_\beta x(\beta)\gamma_1(\beta-\tau)}{|\Delta x|^2|\partial_\beta x(\beta)|^2}\Big)d\tau d\beta,$$

as before we find $G^1_{1,33}\leq C\|z\|_{H^1}$. In the same way as in $G^1_{1,31}$,

$$G^1_{1,35}=-\frac{1}{\pi}\int_{\mathbb{T}}\int_{\mathbb{R}}Q^2_y(\beta)\frac{(\Delta x)^\perp-\tau\partial_\beta^\perp x(\beta)}{|\Delta x|^4}\cdot\frac{\partial_\beta x(\beta)}{|\partial_\beta x(\beta)|^2}\Delta x\cdot\Delta\partial_\beta z(\beta)\gamma_1(\beta-\tau)d\tau d\beta$$

then, $G^1_{1,35}\leq C\|z\|_{H^1}$. For G^2_1 we descompose in the same way:

$$\begin{aligned}
G^2_{1,1}&=\frac{\alpha+\pi}{2\pi}\int_{\mathbb{T}}\frac{\partial_\beta x(\beta,t)}{|\partial_\beta x(\beta,t)|^2}\cdot\partial_\beta((Q^2_x-Q^2_y)BR(\gamma_2,h)_x)(\beta,t)d\beta;\\
G^2_{1,2}&=\frac{\alpha+\pi}{2\pi}\int_{\mathbb{T}}\frac{\partial_\beta x(\beta,t)}{|\partial_\beta x(\beta,t)|^2}\cdot\partial_\beta Q^2_y(\beta)(BR(\gamma_2,h)_x-BR(\xi_2,h)_y)(\beta,t)d\beta;\\
G^2_{1,3}&=\frac{\alpha+\pi}{2\pi}\int_{\mathbb{T}}\frac{\partial_\beta x(\beta,t)}{|\partial_\beta x(\beta,t)|^2}\cdot Q^2_y(\beta)\partial_\beta(BR(\gamma_2,h)_x-BR(\xi_2,h)_y)(\beta,t)d\beta;\\
G^2_{1,4}&=\frac{\alpha+\pi}{2\pi}\int_{\mathbb{T}}\frac{\partial_\beta z(\beta,t)}{|\partial_\beta x(\beta,t)|^2}\cdot\partial_\beta(Q^2_yBR(\xi_2,h)_y)(\beta,t)d\beta;\\
G^2_{1,5}&=\frac{\alpha+\pi}{2\pi}\int_{\mathbb{T}}\partial_\beta(Q^2_yBR(\xi_2,h)_y)(\beta,t)\cdot\partial_\alpha y(\beta)\Big(\frac{1}{|\partial_\alpha x(\beta)|^2}-\frac{1}{|\partial_\alpha y(\beta)|^2}\Big)d\beta.
\end{aligned}$$

If we integrate by parts,

$$G^2_{1,1}\leq C\|\mathcal{F}(x)\|_{L^\infty}\|\partial^2_\alpha x\|_{L^2}\|Q^2_x-Q^2_y\|_{L^\infty}\|BR(\gamma_2,h)_x\|_{L^2}\leq C\|z\|_{H^1}.$$

Again, with the same procedure as in I^2_2 we get:

$$|G^2_{1,2}|+|G^2_{1,4}|+|G^2_{1,5}|\leq C\|z\|_{H^1}.$$

Then, we have to estimate $G^2_{1,3}$. To do that we write $G^2_{1,3}=G^2_{1,31}+G^2_{1,32}+G^2_{1,33}+G^2_{1,34}+G^2_{1,35}+G^2_{1,36}$:

$$\begin{aligned}
G^2_{1,31}&=\frac{1}{2\pi}\int_{\mathbb{T}}\int_{\mathbb{R}}Q^2_y(\beta)\frac{(\Delta xh)^\perp}{|\Delta xh|^2}\cdot\frac{\partial_\beta x(\beta)}{|\partial_\beta x(\beta)|^2}\partial_\beta w_2(\beta-\tau)d\tau d\beta,\\
G^2_{1,32}&=\frac{1}{2\pi}\int_{\mathbb{T}}\int_{\mathbb{R}}Q^2_y(\beta)\Big(\frac{(\Delta xh)^\perp}{|\Delta xh|^2}-\frac{(\Delta yh)^\perp}{|\Delta yh|^2}\Big)\cdot\frac{\partial_\beta x(\beta)}{|\partial_\beta x(\beta)|^2}\partial_\beta\xi_2(\beta-\tau)d\tau d\beta,\\
G^2_{1,33}&=\frac{1}{2\pi}\int_{\mathbb{T}}\int_{\mathbb{R}}Q^2_y(\beta)\frac{\partial_\beta^\perp z(\beta)}{|\Delta xh|^2}\cdot\frac{\partial_\beta x(\beta)}{|\partial_\beta x(\beta)|^2}\gamma_2(\beta-\tau)d\tau d\beta,\\
G^2_{1,34}&=\frac{1}{2\pi}\int_{\mathbb{T}}\int_{\mathbb{R}}Q^2_y(\beta)(\Delta\partial_\beta yh)^\perp\Big(\frac{\gamma_2(\beta-\tau)}{|\Delta xh|^2}-\frac{\xi_2(\beta-\tau)}{|\Delta yh|^2}\Big)\cdot\frac{\partial_\beta x(\beta)}{|\partial_\beta x(\beta)|^2}d\tau d\beta,
\end{aligned}$$

$$
\begin{aligned}
G_{1,35}^2 &= -\frac{1}{\pi}\int_{\mathbb{T}}\int_{\mathbb{R}} Q_y^2(\beta)\frac{(\Delta x)^{\perp}}{|\Delta x|^4}\cdot\frac{\partial_\beta xh(\beta)}{|\partial_\beta xh(\beta)|^2}\Delta xh\cdot\partial_\beta z(\beta)\gamma_2(\beta-\tau)d\tau d\beta,\\
G_{1,36}^2 &= -\frac{1}{\pi}\int_{\mathbb{T}}\int_{\mathbb{R}} \frac{Q_y^2(\beta)\partial_\beta x(\beta)}{|\partial_\beta x(\beta)|^2}\cdot(\frac{(\Delta xh)^{\perp}\Delta xh\cdot\Delta\partial_\beta yh}{|\Delta xh|^4}\gamma_2(\beta-\tau)\\
&-\frac{(\Delta yh)^{\perp}\Delta yh\cdot\Delta\partial_\beta yh}{|\Delta yh|^4}\xi_2(\beta-\tau))d\tau d\beta.
\end{aligned}
$$

Since

$$
\frac{\gamma_2(\beta-\tau)}{|\Delta xh|^2}-\frac{\xi_2(\beta-\tau)}{|\Delta yh|^2}=\frac{z^{\perp}}{|\Delta xh|^2}+\Delta yh^{\perp}(\frac{|y|^2-|x|^2}{|\Delta xh|^2|\Delta yh|^2})+\Delta yh^{\perp}\frac{2z(\beta)h(\alpha-\beta)}{|\Delta xh|^2|\Delta yh|^2}
$$

then, $G_{1,32}^2\leq C\|z\|_{H^1}$. For the term $G_{1,31}^1$ since we have controlled the L^2-norm of the w, we have to integrate by parts:

$$
\begin{aligned}
G_{1,31}^2 &= -\frac{1}{2\pi}\int_{\mathbb{T}}\int_{\mathbb{R}}\partial_\beta(Q_y^2(\beta)\frac{(\Delta xh)^{\perp}}{|\Delta xh|^2}\cdot\frac{\partial_\beta x(\beta)}{|\partial_\beta x(\beta)|^2})w_2(\beta-\tau)d\tau d\beta\\
&\leq C\|w_2\|_{L^2}\leq C\|z\|_{H^1}.
\end{aligned}
$$

$G_{1,33}^2$ and $G_{1,35}^2$ is estimated directly:

$$
\begin{aligned}
G_{1,33}^2 &\leq C\|d(x,h)\|_{L^\infty}\|\mathcal{F}(x)\|_{L^\infty}^{\frac{1}{2}}\|Q_y^2\|_{L^2}\|z\|_{H^1}\|\gamma_2\|_{L^\infty}\leq C\|z\|_{H^1},\\
G_{1,35}^2 &\leq C\|Q_y^2\|_{L^\infty}\|d(x,h)\|_{L^\infty}\|\partial_\alpha z\|_{L^2}\|\gamma_2\|_{L^2}\leq C\|z\|_{H^1}.
\end{aligned}
$$

For $G_{1,34}^2$, since

$$
\frac{\gamma_2(\beta-\tau)}{|\Delta xh|^2}-\frac{\xi_2(\beta-\tau)}{|\Delta yh|^2}=\frac{w_2}{|\Delta xh|^2}+\xi_2(\frac{|x|^2-|y|^2}{|\Delta xh|^2|\Delta yh|^2})+\xi_2\frac{2z(\beta)\cdot h(\beta-\tau)}{|\Delta xh|^2|\Delta yh|^2}
$$

then,

$$
\begin{aligned}
G_{1,34}^2 &\leq C\|\mathcal{F}\|_{L^\infty}^{\frac{1}{2}}\|Q_y^2\|_{L^\infty}\|d(x,h)\|_{L^\infty}(\|\partial_\alpha y\|_{L^2}+\|\partial_\alpha h\|_{L^2})\|w_2\|_{L^2}\\
&+C\|\mathcal{F}\|_{L^\infty}^{\frac{1}{2}}\|Q_y^2\|_{L^\infty}(\|\partial_\alpha y\|_{L^2}+\|\partial_\alpha h\|_{L^2})\|d(x,h)\|_{L^\infty}\|d(y,h)\|_{L^\infty}\|z\|_{L^2}\|h\|_{L^\infty}\|\xi_2\|_{L^\infty}\\
&\leq C\|z\|_{H^1}.
\end{aligned}
$$

Finally, since:

$$
\begin{aligned}
&\frac{(\Delta xh)^{\perp}\Delta xh\cdot\Delta\partial_\beta yh}{|\Delta xh|^4}\gamma_2(\beta-\tau)-\frac{(\Delta yh)^{\perp}\Delta yh\cdot\Delta\partial_\beta yh}{|\Delta yh|^4}\xi_2(\beta-\tau)\\
&=\frac{(\Delta xh)^{\perp}\Delta xh\cdot\Delta\partial_\beta yh}{|\Delta xh|^4}w_2(\beta-\tau)+\frac{z^{\perp}\Delta xh}{|\Delta xh|^4}\cdot\Delta\partial_\alpha yh\xi_2+\frac{\Delta yh^{\perp}z(\alpha)\cdot\Delta\partial_\alpha yh\xi_2}{|\Delta xh|^4}\\
&+\frac{(|\Delta yh|^2+|\Delta xh|^2)(|x|^2-|y|^2)}{|\Delta xh|^4|\Delta yh|^4}\Delta yh^{\perp}\Delta yh\cdot\Delta\partial_\alpha yh\xi_2\\
&+\frac{(|\Delta yh|^2+|\Delta xh|^2)2z(\alpha)\cdot h(\alpha-\beta)}{|\Delta xh|^4|\Delta yh|^4}\Delta yh^{\perp}\Delta yh\cdot\Delta\partial_\alpha yh\xi_2,
\end{aligned}
$$

we have $G_{1,36}^2\leq C\|z\|_{H^1}$. The same decomposition is going to work to control G_2 in the same way than above. The analogous to the term $G^1_{1,33}$:

$$G^1_{2,33} = \frac{1}{2\pi}\int_{-\pi}^{\alpha} Q_y^2(\beta)\int_{\mathbb{R}} \frac{(\Delta\partial_\beta z)^\perp}{|\Delta x|^2}\cdot\frac{\partial_\beta x(\beta)}{|\partial_\beta x(\beta)|^2}\gamma_1(\beta-\tau)d\tau d\beta.$$

Here we can not integrate by parts as in $G^1_{1,33}$. Then, we decompose further $G^1_{2,33} = G^1_{2,331} + G^1_{2,332}$:

$$\begin{aligned}
G^1_{2,331} &= \frac{1}{2\pi}\int_{-\pi}^{\alpha} Q_y^2(\beta)\frac{\partial_\beta x(\beta)}{|\partial_\beta x(\beta)|^2}\cdot\int_{\mathbb{R}}(\Delta\partial_\beta z)^\perp(\frac{\gamma_1(\beta-\tau)}{|\Delta x|^2} - \frac{\gamma_1(\beta)}{\tau^2|\partial_\beta x(\beta)|^2})d\tau d\beta,\\
G^1_{2,332} &= \frac{1}{2}\int_{-\pi}^{\alpha} Q_y^2(\beta)\gamma_1(\beta)\frac{\partial_\beta x(\beta)}{|\partial_\beta x(\beta)|^4}\cdot\Lambda(\partial_\beta^\perp z)(\beta)d\beta.
\end{aligned}$$

The term $G^1_{2,331}$ can be estimated in the same manner as we have been used before. But we have to be careful with $G^1_{2,332}$.

First of all we add $G^1_{2,332} + G$ where:

$$G = -\frac{\alpha+\pi}{4\pi}\int_{\mathbb{T}} Q_y^2(\beta)\gamma_1(\beta)\frac{\partial_\beta x(\beta)}{|\partial_\beta x(\beta)|^4}\cdot\Lambda(\partial_\beta^\perp z)(\beta)d\beta.$$

That G allow us to conclude that $(G^1_{2,332}+G)(\pi) = (G^1_{2,332}+G)(-\pi) = 0$.

Using integration by parts on Λ we get $|G| \leq C\|z\|_{H^1}$, then the extra term which appears when we add G is controlled.

Since G^1_2 is a part of the term I_3, we are going to estimate

$$\int_{\mathbb{T}} z\cdot\partial_\alpha x(G^1_{2,332}+G)d\alpha$$

to finish the estimates on G^1_2.

Here we use the fact that

$$z(\alpha)\cdot\partial_\alpha x(\alpha) = \partial_\alpha(\int_{-\pi}^{\alpha} z(\beta)\cdot\partial_\beta x(\beta)d\beta).$$

Integratinng by parts we split

$$\int_{\mathbb{T}} z\cdot\partial_\alpha x(G^1_{2,332}+G)d\alpha = I + II$$

where

$$\begin{aligned}
I &= -\frac{1}{2}\int_{\mathbb{T}}\int_{-\pi}^{\alpha} z(\beta)\cdot\partial_\beta x(\beta)d\beta Q_y^2(\alpha)\gamma_1(\alpha)\frac{\partial_\beta x(\alpha)}{|\partial_\beta x(\alpha)|^4}\cdot\Lambda(\partial_\beta^\perp z)(\alpha)d\alpha,\\
II &= \frac{1}{4\pi}\int_{\mathbb{T}}\int_{-\pi}^{\alpha} z(\beta)\cdot\partial_\beta x(\beta)d\beta d\alpha\int_{\mathbb{T}} Q_y^2(\beta)\gamma_1(\beta)\frac{\partial_\beta x(\beta)}{|\partial_\beta x(\beta)|^4}\cdot\Lambda(\partial_\beta^\perp z)(\beta)d\beta.
\end{aligned}$$

Using integration by parts on Λ, it is easy to see that $|II| \leq C\|z\|_{H^1}$. Similary for I, if we call $a(\alpha) = \int_{-\pi}^{\alpha} z(\beta)\cdot\partial_\beta x(\beta)d\beta$:

$$\begin{aligned}
I &= -\frac{1}{2}\int_{\mathbb{T}} a(\alpha)Q_y^2(\alpha)\gamma_1(\alpha)\frac{\partial_\alpha x(\alpha)}{|\partial_\alpha x(\alpha)|^4}\cdot\Lambda(\partial_\alpha^\perp z)(\alpha)d\alpha\\
&= -\frac{1}{2}\int_{\mathbb{T}}\Lambda(aQ_y^2\gamma_1\frac{\partial_\alpha x}{|\partial_\alpha x|^4})(\alpha)\cdot\partial_\alpha^\perp z(\alpha)d\alpha.
\end{aligned}$$

Since $\|\Lambda f\|_{L^2} = \|H(\partial_\alpha f)\|_{L^2} \leq C\|\partial_\alpha f\|_{L^2}$,

$$I \leq C\|z\|_{H^1}^2$$

In order to finish the estimates on I_3 we will estimate G_2^2. We decompose it as follows:

$$\begin{aligned}
G_{2,1}^2 &= -\int_{-\pi}^{\alpha} \frac{\partial_\beta x(\beta,t)}{|\partial_\beta x(\beta,t)|^2} \cdot \partial_\beta((Q_x^2 - Q_y^2)BR(\gamma_2,h)_x)(\beta,t)d\beta;\\
G_{2,2}^2 &= -\int_{-\pi}^{\alpha} \frac{\partial_\beta x(\beta,t)}{|\partial_\beta x(\beta,t)|^2} \cdot \partial_\beta Q_y^2(\beta)(BR(\gamma_2,h)_x - BR(\xi_2,h)_y)(\beta,t)d\beta;\\
G_{2,3}^2 &= -\int_{-\pi}^{\alpha} \frac{\partial_\beta x(\beta,t)}{|\partial_\beta x(\beta,t)|^2} \cdot Q_y^2(\beta)\partial_\beta(BR(\gamma_2,h)_x - BR(\xi_2,h)_y)(\beta,t)d\beta;\\
G_{2,4}^2 &= -\int_{-\pi}^{\alpha} \frac{\partial_\beta z(\beta,t)}{|\partial_\beta x(\beta,t)|^2} \cdot \partial_\beta(Q_y^2 BR(\xi_2,h)_y)(\beta,t)d\beta;\\
G_{2,5}^2 &= -\int_{-\pi}^{\alpha} \partial_\beta(Q_y^2 BR(\xi_2,h)_y)(\beta,t)\cdot\partial_\alpha y(\beta)(\frac{1}{|\partial_\alpha x(\beta)|^2} - \frac{1}{|\partial_\alpha y(\beta)|^2})d\beta.
\end{aligned}$$

All terms where we do not need to integrate by parts are estimated in the same way as in G_1^2, therefore we only observe what happen with $G_{2,31}^2$.

$$\begin{aligned}
G_{2,31}^2 &= -\frac{1}{2\pi}\int_{\pi}^{\alpha}\int_{\mathbb{R}} Q_y^2(\beta)\frac{(\Delta xh)^\perp}{|\Delta xh|^2}\cdot\frac{\partial_\beta x(\beta)}{|\partial_\beta x(\beta)|^2}\partial_\beta w_2(\beta-\tau)d\tau d\beta\\
&= \frac{1}{2\pi}\int_{\pi}^{\alpha} Q_y^2(\beta)\frac{\partial_\beta x(\beta)}{|\partial_\beta x(\beta)|^2}\cdot\int_{\mathbb{R}}\frac{(\Delta xh)^\perp}{|\Delta xh|^2}\partial_\tau w_2(\beta-\tau)d\tau d\beta\\
&= \frac{1}{2\pi}\int_{\pi}^{\alpha} Q_y^2(\beta)\frac{\partial_\beta x(\beta)}{|\partial_\beta x(\beta)|^2}\cdot\int_{\mathbb{R}}\partial_\tau(\frac{(\Delta xh)^\perp}{|\Delta xh|^2})w_2(\beta-\tau)d\tau d\beta,
\end{aligned}$$

since

$$\partial_\tau(\frac{(\Delta xh)^\perp}{|\Delta xh|^2}) = \frac{\partial_\beta^\perp h(\beta-\tau)}{|\Delta xh|^2} - \frac{2\Delta xh^\perp\Delta xh\cdot\partial_\beta h(\beta-\tau)}{|\Delta xh|^4}$$

then,

$$G_{2,31}^2 \leq C\|z\|_{H^1}^2.$$

Putting all above estimates together we get:

$$I_3 \leq C\|z\|_{H^1}.$$

Thus, we have estimated the L^2-norm,

$$\frac{1}{2}\frac{d}{dt}\|z\|_{L^2}^2 \leq C\|z\|_{H^1}^2.$$

The next step is estimate:

$$\frac{1}{2}\frac{d}{dt}\|\partial_\alpha z\|_{L^2}^2 = \int_{\mathbb{T}}\partial_\alpha z\cdot\partial_\alpha z_t d\alpha = I_5 + I_6 + I_7 + I_8 + I_9$$

where

$$
\begin{aligned}
I_5 &= \int_{\mathbb{T}} \partial_\alpha z \cdot \partial_\alpha Q_x^2 (BR(\gamma_1, x)_x + BR(\gamma_2, h)_x - BR(\xi_1, y)_y - BR(\xi_2, h)_y) d\alpha, \\
I_6 &= \int_{\mathbb{T}} \partial_\alpha z \cdot Q_x^2 \partial_\alpha (BR(\gamma_1, x)_x + BR(\gamma_2, h)_x - BR(\xi_1, y)_y - BR(\xi_2, h)_y) d\alpha, \\
I_7 &= \int_{\mathbb{T}} \partial_\alpha z \cdot \partial_\alpha (Q_x^2 - Q_y^2)(BR(\xi_1, y)_y + BR(\xi_2, h)_y) d\alpha, \\
I_8 &= \int_{\mathbb{T}} \partial_\alpha z \cdot \partial_\alpha ((c_x - c_y)\partial_\alpha x) d\alpha, \\
I_9 &= \int_{\mathbb{T}} \partial_\alpha z \cdot \partial_\alpha (c_y \partial_\alpha z) d\alpha.
\end{aligned}
$$

We left to the end the estimation of the term I_6, because we will need looking for the R-T condition to control it. now let us control the remaining terms. Since $\partial_\alpha Q_x^2$ is a "good" term, we can estimate I_5 just like we did with I_2:

$$|I_5| \leq C\|z\|_{H^1}^2.$$

The term I_7 is directly:

$$I_7 \leq C\|\partial_\alpha z\|_{L^2}\|Q_x^2 - Q_y^2\|_{H^1}(\|BR(\xi_1, y)_y\|_{L^\infty} + \|BR(\xi_2, h)_y\|_{L^\infty}) \leq C\|z\|_{H^1}^2.$$

Doing integration by parts and the fact that $|\partial_\alpha x|^2 = A_x(t)$ and $|\partial_\alpha y|^2 = A_y(t)$:

$$
\begin{aligned}
I_8 &= \int_{\mathbb{T}} \partial_\alpha z \cdot \partial_\alpha^2 x (c_x - c_y) d\alpha + \int_{\mathbb{T}} \partial_\alpha z \cdot \partial_\alpha x \partial_\alpha (c_x - c_y) d\alpha \\
&= \int_{\mathbb{T}} \partial_\alpha z \cdot \partial_\alpha^2 x (c_x - c_y) d\alpha - \int_{\mathbb{T}} \partial_\alpha z \cdot \partial_\alpha^2 x (c_x - c_y) d\alpha - \int_{\mathbb{T}} \partial_\alpha^2 z \cdot \partial_\alpha x (c_x - c_y) d\alpha \\
&= -\int_{\mathbb{T}} \partial_\alpha^2 x \cdot \partial_\alpha x (c_x - c_y) d\alpha + \int_{\mathbb{T}} \partial_\alpha^2 y \cdot \partial_\alpha x (c_x - c_y) d\alpha \\
&= -\frac{1}{2}\int_{\mathbb{T}} \partial_\alpha (|\partial_\alpha x|^2)(c_x - c_y) d\alpha + \int_{\mathbb{T}} \partial_\alpha^2 y \cdot (\partial_\alpha x - \partial_\alpha y)(c_x - c_y) d\alpha \\
&+ \frac{1}{2}\int_{\mathbb{T}} \partial_\alpha (|\partial_\alpha y|^2)(c_x - c_y) d\alpha = \int_{\mathbb{T}} \partial_\alpha^2 y \cdot \partial_\alpha z (c_x - c_y) d\alpha,
\end{aligned}
$$

this integral can be estimate in the same manner as I_3. Then, $I_8 \leq C\|z\|_{H^1}^2$.

In order to estimate I_9, we integrate by parts:

$$I_9 = -\int_{\mathbb{T}} \partial_\alpha^2 z \cdot \partial_\alpha z c_y d\alpha = -\frac{1}{2}\int_{\mathbb{T}} \partial_\alpha (|\partial_\alpha z|^2) c_y d\alpha = \frac{1}{2}\int_{\mathbb{T}} |\partial_\alpha z|^2 \partial_\alpha c_y d\alpha,$$

since

$$\|\partial_\alpha c_y\|_{L^\infty} \leq C\|\mathcal{F}(y)\|_{L^\infty}^{\frac{1}{2}}(\|\partial_\alpha BR(\xi_1, y)_y\|_{L^\infty} + \|\partial_\alpha BR(\xi_2, h)_y\|_{L^\infty})$$

and

$$I_9 \leq C\|\partial_\alpha z\|_{L^2}^2\|\partial_\alpha c_y\|_{L^\infty}$$

then, $I_9 \le C\|z\|^2_{H^1}$.

Let us estimate the last teerm I_6. To do that we split $I_6 = I_6^1 + I_6^2$:

$$I_6^1 = \int_{\mathbb{T}} \partial_\alpha z \cdot Q_x^2 \partial_\alpha (BR(\gamma_1, x)_x - BR(\xi_1, y)_y) d\alpha,$$
$$I_6^2 = \int_{\mathbb{T}} \partial_\alpha z \cdot Q_x^2 \partial_\alpha (BR(\gamma_2, h)_x - BR(\xi_2, h)_y) d\alpha.$$

For the study of the I_6^1 we descompose it in the following terms:

$$I_{6,1}^1 = \frac{1}{2\pi}\int_{\mathbb{T}}\int_{\mathbb{R}} Q_x^2(\alpha)\partial_\alpha z(\alpha)\cdot \frac{\Delta x^\perp}{|\Delta x|^2}\partial_\alpha w_1(\alpha-\beta)d\beta d\alpha,$$
$$I_{6,2}^1 = \frac{1}{2\pi}\int_{\mathbb{T}}\int_{\mathbb{R}} Q_x^2(\alpha)\partial_\alpha z(\alpha)\cdot (\frac{(\Delta x)^\perp}{|\Delta x|^2} - \frac{(\Delta y)^\perp}{|\Delta y|^2})\partial_\alpha \xi_1(\alpha-\beta)d\beta d\alpha,$$
$$I_{6,3}^1 = \frac{1}{2\pi}\int_{\mathbb{T}}\int_{\mathbb{R}} Q_x^2(\alpha)\partial_\alpha z(\alpha)\cdot \frac{\Delta \partial_\alpha^\perp z(\alpha)}{|\Delta x|^2}\gamma_1(\alpha-\beta)d\beta d\alpha,$$
$$I_{6,4}^1 = \frac{1}{2\pi}\int_{\mathbb{T}}\int_{\mathbb{R}} Q_x^2(\alpha)\partial_\alpha z(\alpha)\cdot \Delta\partial_\alpha^\perp y(\frac{\gamma_1(\alpha-\beta)}{|\Delta x|^2} - \frac{\xi_1(\alpha-\beta)}{|\Delta y|^2})d\beta d\alpha,$$
$$I_{6,5}^1 = -\frac{1}{\pi}\int_{\mathbb{T}}\int_{\mathbb{R}} Q_x^2(\alpha)\partial_\alpha z(\alpha)\cdot \frac{(\Delta x)^\perp \Delta x\cdot\Delta\partial_\alpha z}{|\Delta x|^4}\gamma_1(\alpha-\beta)d\beta d\alpha,$$
$$I_{6,6}^1 = -\frac{1}{\pi}\int_{\mathbb{T}}\int_{\mathbb{R}} Q_x^2(\alpha)\partial_\alpha z(\alpha)\cdot (\frac{(\Delta x)^\perp \Delta x\cdot\Delta\partial_\alpha y}{|\Delta x|^4}\gamma_1(\alpha-\beta) - \frac{(\Delta y)^\perp \Delta y\cdot\Delta\partial_\alpha y}{|\Delta y|^4}\xi_1(\alpha-\beta))d\beta d\alpha.$$

We proceed as before to obtain:

$$\sum_{i=2}^{6} I_{6,i}^1 \le C\|z\|^2_{H^1}.$$

We will study $I_{6,1}^1$ in which we will find the R-T condition. We split

$$\begin{aligned}
I_{6,1}^1 &= \frac{1}{2\pi}\int_{\mathbb{T}}\int_{\mathbb{R}} Q_x^2(\alpha)\partial_\alpha z(\alpha)\cdot (\frac{\Delta x^\perp}{|\Delta x|^2} - \frac{\partial_\alpha^\perp x(\alpha)}{\beta|\partial_\alpha x(\alpha)|^2})\partial_\alpha w_1(\alpha-\beta)d\beta d\alpha\\
&+ \frac{1}{2\pi}\int_{\mathbb{T}}\int_{\mathbb{R}} Q_x^2(\alpha)\partial_\alpha z(\alpha)\cdot \frac{\partial_\alpha^\perp x(\alpha)}{\beta|\partial_\alpha x(\alpha)|^2}\partial_\alpha w_1(\alpha-\beta)d\beta d\alpha\\
&\le C\|z\|^2_{H^1} + \frac{1}{2}\int_{\mathbb{T}} Q_x^2(\alpha)\partial_\alpha z(\alpha)\cdot\frac{\partial_\alpha^\perp x(\alpha)}{|\partial_\alpha x(\alpha)|^2}H(\partial_\alpha w_1)(\alpha)d\alpha.
\end{aligned}$$

Let call $I_{6,11}^1$ to the last integral. In order to simplify the notation we are going to use $B_x = BR(\gamma_1, x)_x + BR(\gamma_2, h)_x$ and $B_y = BR(\xi_1, y)_y + BR(\xi_2, h)_y$.

Since

$$w_1 = -2B_x\cdot\partial_\alpha x + 2B_y\cdot\partial_\alpha y - 2\kappa^1\frac{\rho^2}{\mu^2}(\nabla P_2^{-1}(x)\cdot\partial_\alpha x - \nabla P_2^{-1}(y)\cdot\partial_\alpha y),$$

we check that:

$$\begin{aligned}
I_{6,11}^1 &= \frac{1}{2}\int_{\mathbb{T}} Q_x^2(\alpha)\partial_\alpha z(\alpha)\cdot\frac{\partial_\alpha^\perp x(\alpha)}{|\partial_\alpha x(\alpha)|^2}\Lambda(w_1)(\alpha)d\alpha = \frac{1}{2}\int_{\mathbb{T}}\Lambda(Q_x^2\partial_\alpha z\cdot\frac{\partial_\alpha^\perp x}{|\partial_\alpha x|^2})(\alpha)w_1(\alpha)d\alpha\\
&= \frac{1}{2}\int_{\mathbb{T}}\Lambda(Q_x^2\partial_\alpha z\cdot\frac{\partial_\alpha^\perp x}{|\partial_\alpha x|^2})(\alpha)(-2B_x\cdot\partial_\alpha x + 2B_y\cdot\partial_\alpha y)d\alpha
\end{aligned}$$

$$+\frac{K}{2}\int_{\mathbb{T}}\Lambda(Q_x^2\partial_\alpha z\cdot\frac{\partial_\alpha^\perp x}{|\partial_\alpha x|^2})(\alpha)(\nabla P_2^{-1}(x)\cdot\partial_\alpha x-\nabla P_2^{-1}(y)\cdot\partial_\alpha y)d\alpha\equiv J_1+J_2.$$

Now we descompose J_1.

$$J_1^1=\frac{1}{2}\int_{\mathbb{T}}\Lambda(Q_x^2\partial_\alpha z\cdot\frac{\partial_\alpha^\perp x}{|\partial_\alpha x|^2})(\alpha)(-2B_x\cdot\partial_\alpha z(\alpha))d\alpha,$$
$$J_1^2=\frac{1}{2}\int_{\mathbb{T}}\Lambda(Q_x^2\partial_\alpha z\cdot\frac{\partial_\alpha^\perp x}{|\partial_\alpha x|^2})(\alpha)(2(B_y-B_x)\cdot\partial_\alpha y)d\alpha.$$

To study J_1^2 we have not introduce any new idea:

$$\begin{aligned}J_1^2&=\frac{1}{2}\int_{\mathbb{T}}Q_x^2(\alpha)\partial_\alpha z(\alpha)\cdot\frac{\partial_\alpha^\perp x(\alpha)}{|\partial_\alpha x(\alpha)|^2}\Lambda(2(B_y-B_x)\cdot\partial_\alpha y)(\alpha)d\alpha\\
&=\frac{1}{2}\int_{\mathbb{T}}Q_x^2(\alpha)\partial_\alpha z(\alpha)\cdot\frac{\partial_\alpha^\perp x(\alpha)}{|\partial_\alpha x(\alpha)|^2}(\Lambda(2(B_y-B_x)\cdot\partial_\alpha y)(\alpha)-2(B_y-B_x)\cdot\Lambda(\partial_\alpha y)(\alpha))d\alpha\\
&+\frac{1}{2}\int_{\mathbb{T}}Q_x^2(\alpha)\partial_\alpha z(\alpha)\cdot\frac{\partial_\alpha^\perp x(\alpha)}{|\partial_\alpha x(\alpha)|^2}2(B_y-B_x)\cdot H(\partial_\alpha y)(\alpha)d\alpha\le C\|z\|_{H^1}^2.\end{aligned}$$

For J_1^1:

$$J_{1,1}^1=\frac{1}{A_x(t)}\int_{\mathbb{T}}\Lambda(Q_x^2\partial_\alpha z_1\partial_\alpha x_2)(\alpha)B_x^1\partial_\alpha z_1(\alpha)d\alpha,$$
$$J_{1,2}^1=\frac{1}{A_x(t)}\int_{\mathbb{T}}\Lambda(Q_x^2\partial_\alpha z_1\partial_\alpha x_2)(\alpha)B_x^2\partial_\alpha z_2(\alpha)d\alpha,$$
$$J_{1,3}^1=-\frac{1}{A_x(t)}\int_{\mathbb{T}}\Lambda(Q_x^2\partial_\alpha z_2\partial_\alpha x_1)(\alpha)B_x^1\partial_\alpha z_1(\alpha)d\alpha,$$
$$J_{1,4}^1=-\frac{1}{A_x(t)}\int_{\mathbb{T}}\Lambda(Q_x^2\partial_\alpha z_2\partial_\alpha x_1)(\alpha)B_x^2\partial_\alpha z_2(\alpha)d\alpha.$$

where B_x^j for $j=1,2$ are the components of B_x. Using the commutator estimate for Λ:

$$J_{1,1}^1\le C\|z\|_{H^1}^2+\frac{1}{A_x(t)}\int_{\mathbb{T}}Q_x^2B_x^1\partial_\alpha x_2(\alpha)\partial_\alpha z_1(\alpha)\Lambda(\partial_\alpha z_1)(\alpha)d\alpha.$$

For $J_{1,2}^1$ we do the same:

$$J_{1,2}^1\le C\|z\|_{H^1}^2+\frac{1}{A_x(t)}\int_{\mathbb{T}}Q_x^2B_x^2\partial_\alpha x_2(\alpha)\partial_\alpha z_2(\alpha)\Lambda(\partial_\alpha z_1)(\alpha)d\alpha$$

but for the last integral, since $\partial_\alpha x_2\partial_\alpha^2 z_2=-\partial_\alpha x_1\partial_\alpha^2 z_1-\partial_\alpha z\cdot\partial_\alpha^2 y$ we have:

$$\begin{aligned}&\frac{1}{A_x(t)}\int_{\mathbb{T}}Q_x^2B_x^2\partial_\alpha x_2(\alpha)\partial_\alpha z_2(\alpha)\Lambda(\partial_\alpha z_1)(\alpha)d\alpha=\frac{1}{A_x(t)}\int_{\mathbb{T}}\Lambda(Q_x^2B_x^2\partial_\alpha x_2\partial_\alpha z_2)(\alpha)\partial_\alpha z_1(\alpha)d\alpha\\
&\le C\|z\|_{H^1}^2+\frac{1}{A_x(t)}\int_{\mathbb{T}}Q_x^2B_x^2\Lambda(\partial_\alpha x_2\partial_\alpha z_2)(\alpha)\partial_\alpha z_1(\alpha)d\alpha\le C\|z\|_{H^1}^2\\
&+\frac{1}{A_x(t)}\int_{\mathbb{T}}Q_x^2B_x^2H(\partial_\alpha^2 x_2\partial_\alpha z_2)(\alpha)\partial_\alpha z_1(\alpha)d\alpha+\frac{1}{A_x(t)}\int_{\mathbb{T}}Q_x^2B_x^2H(\partial_\alpha x_2\partial_\alpha^2 z_2)(\alpha)\partial_\alpha z_1(\alpha)d\alpha\\
&\le C\|z\|_{H^1}^2+\frac{1}{A_x(t)}\int_{\mathbb{T}}Q_x^2B_x^2H(\partial_\alpha x_2\partial_\alpha^2 z_2)(\alpha)\partial_\alpha z_1(\alpha)d\alpha\le C\|z\|_{H^1}^2\end{aligned}$$

$$
\begin{aligned}
&-\frac{1}{A_x(t)}\int_{\mathbb{T}} Q_x^2 B_x^2 H(\partial_\alpha x_1 \partial_\alpha^2 z_1)(\alpha)\partial_\alpha z_1(\alpha)d\alpha - \frac{1}{A_x(t)}\int_{\mathbb{T}} Q_x^2 B_x^2 H(\partial_\alpha z\cdot\partial_\alpha^2 y)(\alpha)\partial_\alpha z_1(\alpha)d\alpha\\
&\leq C\|z\|_{H^1}^2 - \frac{1}{A_x(t)}\int_{\mathbb{T}} Q_x^2 B_x^2 \partial_\alpha x_1(\alpha)\partial_\alpha z_1(\alpha) H(\partial_\alpha^2 z_1)(\alpha)d\alpha\\
&\leq C\|z\|_{H^1}^2 - \frac{1}{A_x(t)}\int_{\mathbb{T}} Q_x^2 B_x^2 \partial_\alpha x_1(\alpha)\partial_\alpha z_1(\alpha) \Lambda(\partial_\alpha z_1)(\alpha)d\alpha.
\end{aligned}
$$

If we add,

$$
J_{1,1}^1 + J_{1,2}^1 \leq C\|z\|_{H^1}^2 - \frac{1}{A_x(t)}\int_{\mathbb{T}} Q_x^2 B_x\cdot\partial_\alpha^\perp x(\alpha)\partial_\alpha z_1(\alpha)\Lambda(\partial_\alpha z_1)(\alpha)d\alpha.
$$

In the same way for $J_{1,3}^1$ and $J_{1,4}^1$:

$$
J_{1,3}^1 + J_{1,4}^1 \leq C\|z\|_{H^1}^2 - \frac{1}{A_x(t)}\int_{\mathbb{T}} Q_x^2 B_x\cdot\partial_\alpha^\perp x(\alpha)\partial_\alpha z_2(\alpha)\Lambda(\partial_\alpha z_2)(\alpha)d\alpha.
$$

Therefore,

$$
J_1^1 \leq C\|z\|_{H^1}^2 - \frac{1}{A_x(t)}\int_{\mathbb{T}} Q_x^2 B_x\cdot\partial_\alpha^\perp x(\alpha)\partial_\alpha z(\alpha)\cdot\Lambda(\partial_\alpha z)(\alpha)d\alpha.
$$

Since

$$
\nabla P_2^{-1}(x)\cdot\partial_\alpha x - \nabla P_2^{-1}(y)\cdot\partial_\alpha y = \nabla P_2^{-1}(x)\cdot\partial_\alpha z + (\nabla P_2^{-1}(x) - \nabla P_2^{-1}(y))\cdot\partial_\alpha y
$$

we decompose $J_2 = J_2^1 + J_2^2$, where

$$
\begin{aligned}
J_2^1 &= \frac{K}{2}\int_{\mathbb{T}} \Lambda(Q_x^2\partial_\alpha z\cdot\frac{\partial_\alpha^\perp x}{|\partial_\alpha x|^2})(\alpha)\nabla P_2^{-1}(x)\cdot\partial_\alpha z(\alpha)d\alpha,\\
J_2^2 &= \frac{K}{2}\int_{\mathbb{T}} \Lambda(Q_x^2\partial_\alpha z\cdot\frac{\partial_\alpha^\perp x}{|\partial_\alpha x|^2})(\alpha)(\nabla P_2^{-1}(x) - \nabla P_2^{-1}(y))\cdot\partial_\alpha y(\alpha)d\alpha.
\end{aligned}
$$

Using integration by parts on Λ, we get $J_2^2 \leq C\|z\|_{H^1}$. The next step is decompose J_2^1 in components:

$$
\begin{aligned}
J_{2,1}^1 &= \frac{K}{2A_x(t)}\int_{\mathbb{T}} \Lambda(Q_x^2\partial_\alpha z_1\partial_\alpha x_2)(\alpha)\nabla P_2^{-1}(x)^1\partial_\alpha z_1(\alpha)d\alpha,\\
J_{2,2}^1 &= \frac{K}{2A_x(t)}\int_{\mathbb{T}} \Lambda(Q_x^2\partial_\alpha z_1\partial_\alpha x_2)(\alpha)\nabla P_2^{-1}(x)^2\partial_\alpha z_2(\alpha)d\alpha,\\
J_{2,3}^1 &= -\frac{K}{2A_x(t)}\int_{\mathbb{T}} \Lambda(Q_x^2\partial_\alpha z_2\partial_\alpha x_1)(\alpha)\nabla P_2^{-1}(x)^1\partial_\alpha z_1(\alpha)d\alpha,\\
J_{2,4}^1 &= -\frac{K}{2A_x(t)}\int_{\mathbb{T}} \Lambda(Q_x^2\partial_\alpha z_2\partial_\alpha x_1)(\alpha)\nabla P_2^{-1}(x)^2\partial_\alpha z_2(\alpha)d\alpha.
\end{aligned}
$$

Here we can use the same technique than in J_1^1, then:

$$
J_2^1 \leq C\|z\|_{H^1}^2 - \frac{K}{2A_x(t)}\int_{\mathbb{T}} Q_x^2\nabla P_2^{-1}(x)\cdot\partial_\alpha^\perp x(\alpha)\partial_\alpha z(\alpha)\cdot\Lambda(\partial_\alpha z)(\alpha)d\alpha.
$$

Thus,

$$I^1_{6,1} \leq C\|z\|^2_{H^1} - \frac{2\mu^2}{\kappa^1}\int_{\mathbb{T}} Q^2_x \sigma_x(\alpha)\partial_\alpha z(\alpha)\cdot\Lambda(\partial_\alpha z)(\alpha)d\alpha.$$

The positivity of σ_x give us $I^1_{6,1} \leq C\|z\|^2_{H^1}$.

The remaining term is I^2_6. Since $\partial_\alpha z\cdot\partial^\perp_\alpha z = 0$ we decompose $I^2_6 = \sum_{i=1}^5 I^2_{6,i}$:

$$\begin{aligned}
I^2_{6,1} &= \frac{1}{2\pi}\int_{\mathbb{T}}\int_{\mathbb{R}} Q^2_x(\alpha)\partial_\alpha z(\alpha)\cdot\frac{\Delta xh^\perp}{|\Delta xh|^2}\partial_\alpha w_2(\alpha-\beta)d\beta d\alpha,\\
I^2_{6,2} &= \frac{1}{2\pi}\int_{\mathbb{T}}\int_{\mathbb{R}} Q^2_x(\alpha)\partial_\alpha z(\alpha)\cdot(\frac{(\Delta xh)^\perp}{|\Delta xh|^2} - \frac{(\Delta yh)^\perp}{|\Delta yh|^2})\partial_\alpha \xi_2(\alpha-\beta)d\beta d\alpha,\\
I^2_{6,3} &= \frac{1}{2\pi}\int_{\mathbb{T}}\int_{\mathbb{R}} Q^2_x(\alpha)\partial_\alpha z(\alpha)\cdot\Delta\partial^\perp_\alpha yh(\frac{\gamma_2(\alpha-\beta)}{|\Delta xh|^2} - \frac{\xi_2(\alpha-\beta)}{|\Delta yh|^2})d\beta d\alpha,\\
I^2_{6,4} &= -\frac{1}{\pi}\int_{\mathbb{T}}\int_{\mathbb{R}} Q^2_x(\alpha)\partial_\alpha z(\alpha)\cdot\frac{(\Delta xh)^\perp\Delta xh\cdot\partial_\alpha z(\alpha)}{|\Delta xh|^4}\gamma_2(\alpha-\beta)d\beta d\alpha,\\
I^2_{6,5} &= -\frac{1}{\pi}\int_{\mathbb{T}}\int_{\mathbb{R}} Q^2_x(\alpha)\partial_\alpha z(\alpha)\cdot(\frac{(\Delta xh)^\perp\Delta xh\cdot\Delta\partial_\alpha yh}{|\Delta xh|^4}\gamma_2(\alpha-\beta)\\
&- \frac{(\Delta yh)^\perp\Delta yh\cdot\Delta\partial_\alpha yh}{|\Delta yh|^4}\xi_2(\alpha-\beta))d\beta d\alpha.
\end{aligned}$$

In the same way as in $G^2_{1,32}$, $G^2_{1,34}$ and $G^2_{1,36}$ it is easy to se that

$$|I^2_{6,2}| + |I^2_{6,3}| + |I^2_{6,5}| \leq C\|z\|^2_{H^1}.$$

Directly,

$$I^2_{6,4} \leq C\|d(x,h)\|_{L^\infty}\|Q^2_x\|_{L^\infty}\|\xi_2\|_{L^\infty}\|\partial_\alpha z\|^2_{L^2} \leq C\|z\|^2_{H^1}.$$

Finally,

$$\begin{aligned}
I^2_{6,1} &= \frac{1}{2\pi}\int_{\mathbb{T}}\int_{\mathbb{R}} Q^2_x(\alpha)\partial_\alpha z(\alpha)\cdot\partial_\beta(\frac{\Delta xh^\perp}{|\Delta xh|^2})w_2(\alpha-\beta)d\beta d\alpha\\
&= \frac{1}{2\pi}\int_{\mathbb{T}}\int_{\mathbb{R}} Q^2_x(\alpha)\partial_\alpha z(\alpha)\cdot\frac{\partial^\perp_\alpha h(\alpha-\beta)}{|\Delta xh|^2}w_2(\alpha-\beta)d\beta d\alpha\\
&- \frac{1}{\pi}\int_{\mathbb{T}}\int_{\mathbb{R}} Q^2_x(\alpha)\partial_\alpha z(\alpha)\cdot\frac{\Delta xh^\perp\Delta xh\cdot\partial_\alpha h(\alpha-\beta)}{|\Delta xh|^4}w_2(\alpha-\beta)d\beta d\alpha\\
&\leq C\|\partial_\alpha h\|_{L^\infty}\|d(x,h)\|_{L^\infty}\|Q^2_x\|_{L^\infty}\|\partial_\alpha z\|_{H^1}\|w_2\|_{L^2} \leq C\|z\|^2_{H^1}
\end{aligned}$$

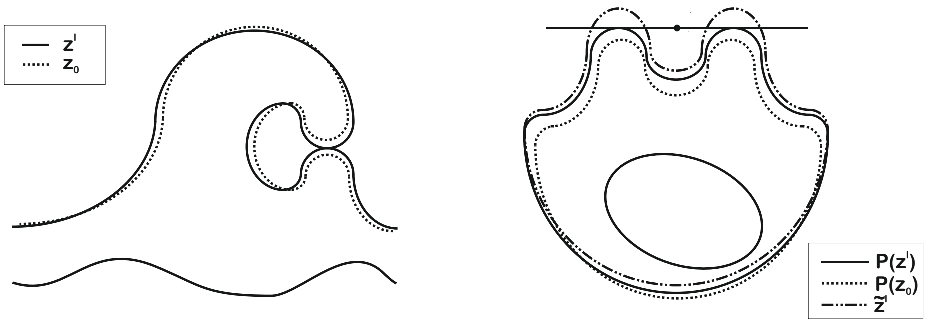
In conclusion,

$$\frac{d}{dt}\|\partial_\alpha z\|^2_{L^2} \leq C\|z\|^2_{H^1}.$$

Thus we get the desired estimate. ■

3.2.4 Existence of the splash singularity

Finally , in this subsection we will apply a perturbative argument to conclude the proof of the main theorem. See the orientative figure 3.2:



(a) Ω (b) $\tilde{\Omega}$

Figure 3.2: Pertubative argument

We start considering a curve z^l as we have defined in subsection 3.2.1. If we apply the conformal map P to this curve, $P(z^l(\alpha))$, and we consider it like an initial data for the transformed Muskat problem, we can assert that there exists a solution $\tilde{z}^l \in \mathcal{C}([0,T], H^3)$ by using subsection 3.2.2.

The next step is consider a perturbation of $z^l(\alpha)$, which we call $z_0(\alpha)$, for which the R-T condition and the arc-chord condition holds. If we take $P(z_0(\alpha))$ as an initial data for the transformed Muskat problem, we get a solution $\tilde{z} \in \mathcal{C}([0,T], H^3)$.

The stability result obtained in subsection 3.2.3 give us the fact that the distance of these two curves is as small as we want, then we have time of existence in between of both.

Since P^{-1} is well defined for $\tilde{z}$ and z^l self-intersects at a point, we can conclude that in the evolution of $z = P^{-1}(\tilde{z})$ there exists a finite time such that z has to break down into a splash singularity.

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